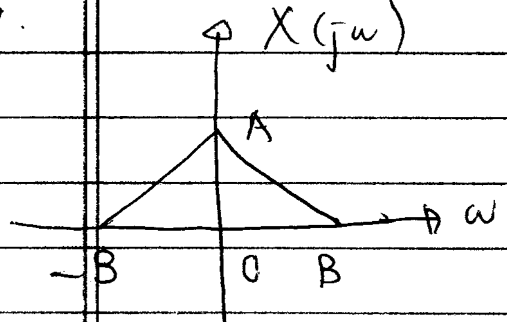
Sampling and reconstruction of bandlimited CT signals

Consider a CT bandlimited signal $x(t)$ with bandwidth B so that

$$X(j\omega) = 0 \quad \text{for } |\omega| \geq B \qquad (1)$$

e.g.

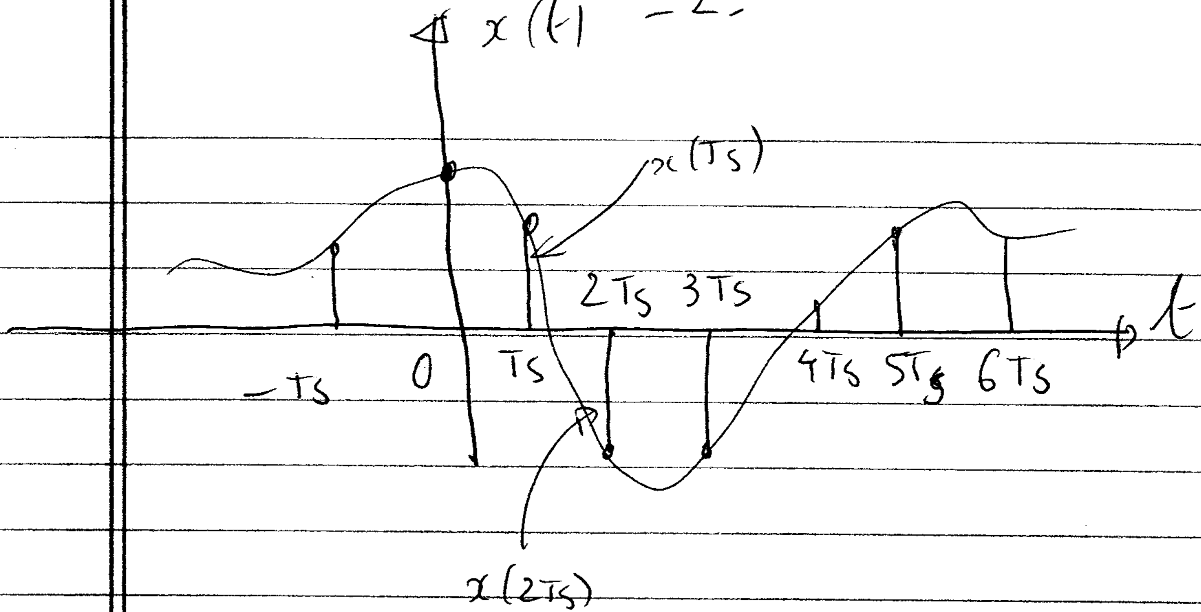


where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt = \text{CT FT of } x(t).$$

~~Aso~~ A very important result which underpins all of DSP and consumer ~~tools~~ products such as CDs, DVD's, digital TV, digital communications, etc... is that $x(t)$ can be reconstructed <u>exactly</u> from its samples $x(nT_s)$ provided $T_s < T_N = \frac{\pi}{B}$

Nyquist sampling period



or equivalently provided

$$\omega_s = \frac{2\pi}{T_s} = \text{sampling frequency} > \frac{2\pi}{T_N} = 2B = \omega_N \quad (2)$$

$= \omega_N$ = Nyquist sampling frequency

This important result is known as the Nyquist sampling theorem. One formula for reconstructing $x(t)$ from its samples is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \underbrace{\frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s}}_{h(t-nT_s)} \quad (3)$$

with

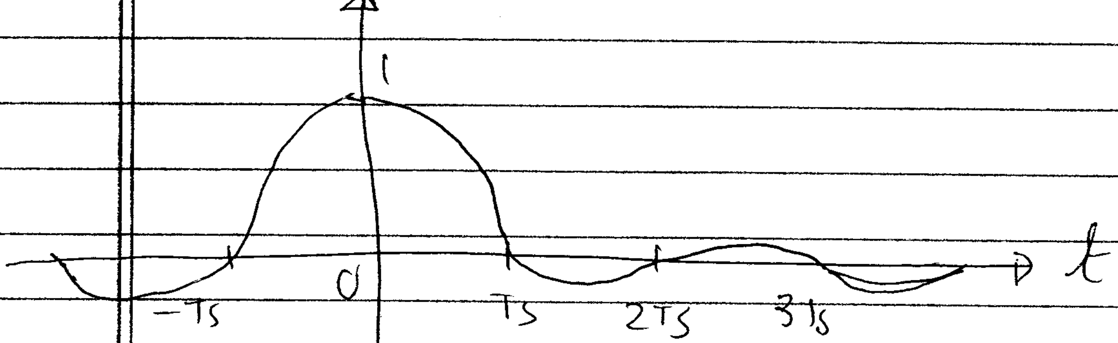
$$h(t) = \frac{\sin(\pi t/T_S)}{\pi t/T_S} = \text{interpolating function} \tag{4}$$

but we shall see later that when ~~T_S~~ $T_S < T_N$ say $T_S = 0.9T_N$ or $0.8T_N$, other choices of interpolating functions are possible.

Note that the interpolation function $h(t)$ is such that

$$h(0) = 1$$

$$h(nT_S) = \frac{\sin(\pi n \cancel{T_S/T_S})}{\pi n T_S/T_S} = 0 \quad \text{for } n \neq 0 \tag{5}$$



so clearly by setting $t = kT_S$ in (3) we find $x(kT_S) = x(kT_S)$.

i.e. the interpolated function $x(t)$ evaluated at the sampling times kT_s ~~gt~~ coincides with the corresponding DT samples.

To analyze the sampling and interpolation process, it is convenient to model CT sampling as follows

$$
\begin{array}{ccccc}
x(t) & \longrightarrow & \otimes & \longrightarrow & z(t) \\
 & & \uparrow & & \\
 & & s(t) & &
\end{array}
$$

$$(6) \quad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \text{train of impulses located at } 0, \pm T_s, \pm 2T_s \ldots$$

$$= \text{the sampling function.}$$

Then

$$z(t) = s(t)\, x(t) = \sum_{n=-\infty}^{\infty} \underbrace{x(nT_s)}_{\text{sampled values of } x(t)} \delta(t - nT_s) \quad (7)$$

where we have used $x(t)\,\delta(t - nT_s) = x(nT_s)\,\delta(t - nT_s)$

So the information contained in $z(t)$ is equivalent to the information contained in the samples $x(nT_s)$.

Let us now evaluate the CT FT of $z(t)$

~~Evaluation z(t)~~

First evaluation (obvious)

$$FT[z(t)] = Z(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)\, FT[\delta(t-nT_s)]$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \exp(-j\omega nT_s) \qquad (8)$$

(recall $\delta(t) \Leftrightarrow 1$

$\delta(t-t_0) \leftrightarrow \exp(-j\omega t_0)$ here $t_0 = nT_s$)

~~If~~

discrete frequency

$$\bar{X}(e^{j\omega_d}) = \sum_{n=-\infty}^{\infty} x(nT_s) \exp(-j\omega_d n) \qquad (9)$$

$=$ DT FT of <u>samples</u> $x(nT_s)$

DT frequency

$$Z(j\omega) = \bar{X}(e^{j\omega T_s}) \quad \text{i.e. } \omega_d = \omega T_s$$

CT frequency

(10)

so $Z(j\omega)$ is just a way of representing the DT FT $\bar{X}(e^{j\omega_d})$ of the samples $x(nT_s)$. This justifies the introduction of the ~~strange~~ ~~sampled~~ ~~wea~~ impulsive sampled signal $z(t)$.

Let us consider now a second, more interesting, evaluation of $Z(j\omega)$. Note that since

$$z(t) = s(t)\, x(t) \qquad (11)$$

we have

$$Z(j\omega) = \frac{1}{2\pi} S(j\omega) * X(j\omega) \qquad (12)$$

where $S(j\omega)$ = FT of sampling function $s(t)$

$$= \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \quad \text{with } \omega_s = \frac{2\pi}{T_s} \qquad (13)$$

Lathi Table 4.1

To develop an intuition about (13) note that if a function is impulsive in one domain, it is periodic in the other domain, e.g.

Time — Fourier

$\delta(t-t_0) \Leftrightarrow \exp(-j\omega t_0)$ periodic

impulsive

$\exp(-j\omega_0 t) \Leftrightarrow 2\pi\,\delta(\omega-\omega_0)$

periodic — impulsive

since $s(t)$ is impulsive <u>and</u> periodic

$S(j\omega)$ is impulsive and periodic

(same form as $s(t)$ but $\left\{\begin{array}{l}\text{amplitude } \omega_s \text{ instead of } 1 \\ \text{spacing between } \omega_s \text{ — of } T_s \\ \text{impulses}\end{array}\right.$)

To evaluate (12) note that

$$Z(j\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) * X(j\omega)$$

$$\frac{\omega_s}{2\pi} = \frac{1}{T_s}$$

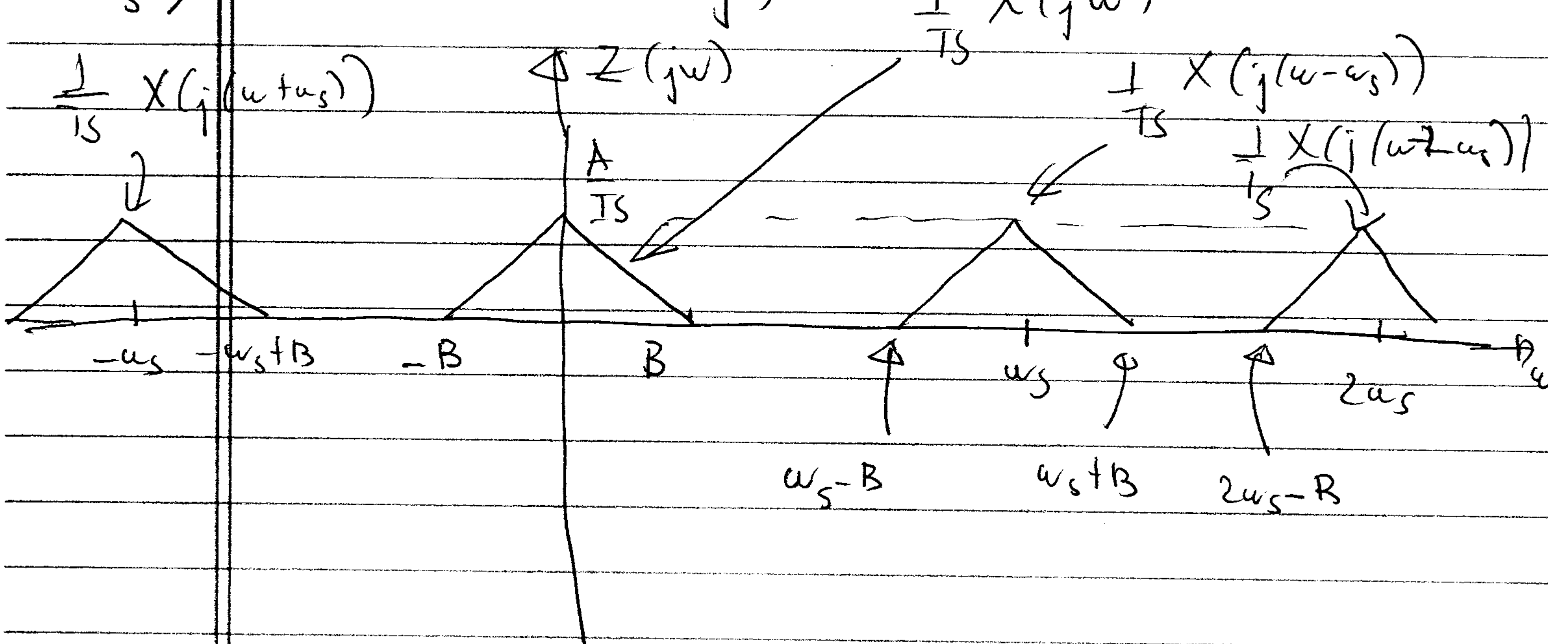
$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \qquad (14)$$

This last expression is the key to understanding sampling

We consider 2 cases:

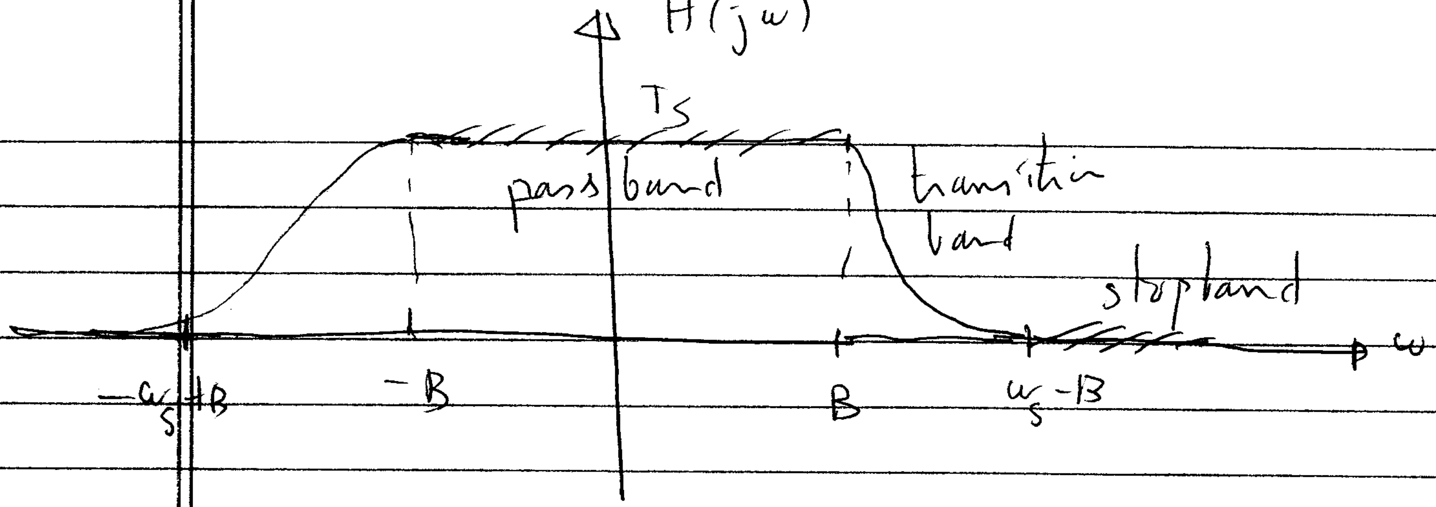
<u>Case 1</u>

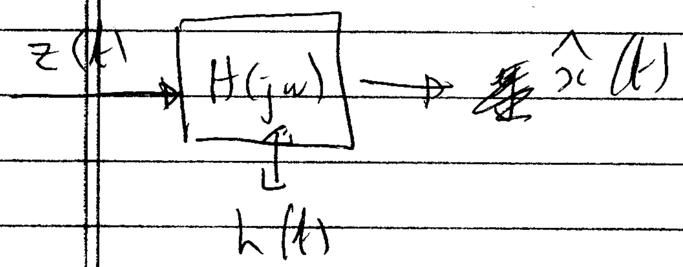
$\omega_s > 2B$ (no aliasing)



Because $\omega_s - B > B$ the copies $\frac{1}{T_s} X(j(\omega - k\omega_s))$ of the ~~FT~~ $X(j\omega)$ of signal $x(t)$ <u>do not overlap</u>. We can therefore recover ~~$X(j\omega)$~~ $x(t)$ by passing $z(t)$ through a ~~lpf~~ lowpass filter $H(j\omega)$ such that

$$H(j\omega) = \begin{cases} T_s & |\omega| \leq B \\ 0 & |\omega| \geq \omega_s - B \end{cases} \qquad (15)$$





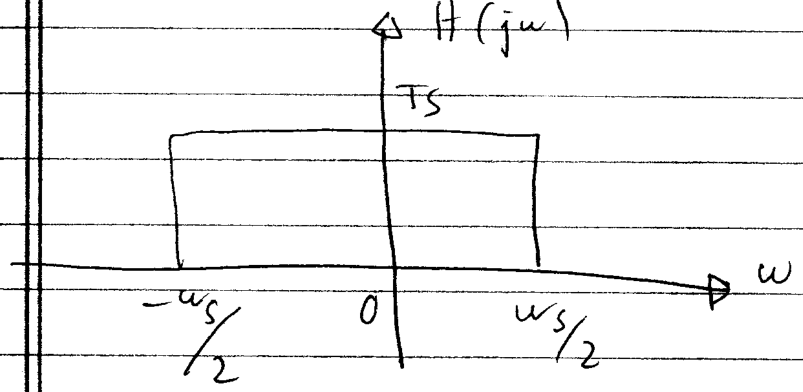
Then $\hat{X}(j\omega) = H(j\omega)\, Z(j\omega) = X(j\omega)$

$$\underbrace{\hat{x}(t)}_{\text{reconstructed signal}} = x(t) \qquad (16)$$

so the signal $x(t)$ is reconstructed <u>exactly</u> from its samples $x(nT_s)$.

Note that <u>any</u> filter satisfying (15) will work.

One filter that does the job is the ideal lowpass filter



$$H(j\omega) = T_S \, rect\left(\frac{\omega}{\omega_S}\right)$$

$$\Longleftrightarrow h(t) = \underbrace{T_S \times \frac{\omega_S}{2\pi}}_{=1} \, sinc\left(\frac{\omega_S}{2} t\right) = \frac{\sin(\pi t / T_S)}{\pi t / T_S} \qquad (17)$$

which is the filter (4)

In this case

$$z(t) \longrightarrow \boxed{h(t)} \longrightarrow x(t)$$

LTI

$$x(t) = h(t) * z(t) = h(t) * \sum_{n=-\infty}^{\infty} x(nT_S)\, \delta(t - nT_S)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_S)\, h(t - nT_S) \qquad (18)$$

which is interpolation formula (3)

The advantage of the ideal lowpass filter is that it has no transition band, so it can be used even if $\omega_s = 2B + \varepsilon$

arbitrary small number

Its disadvantage is that

$$h(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$$

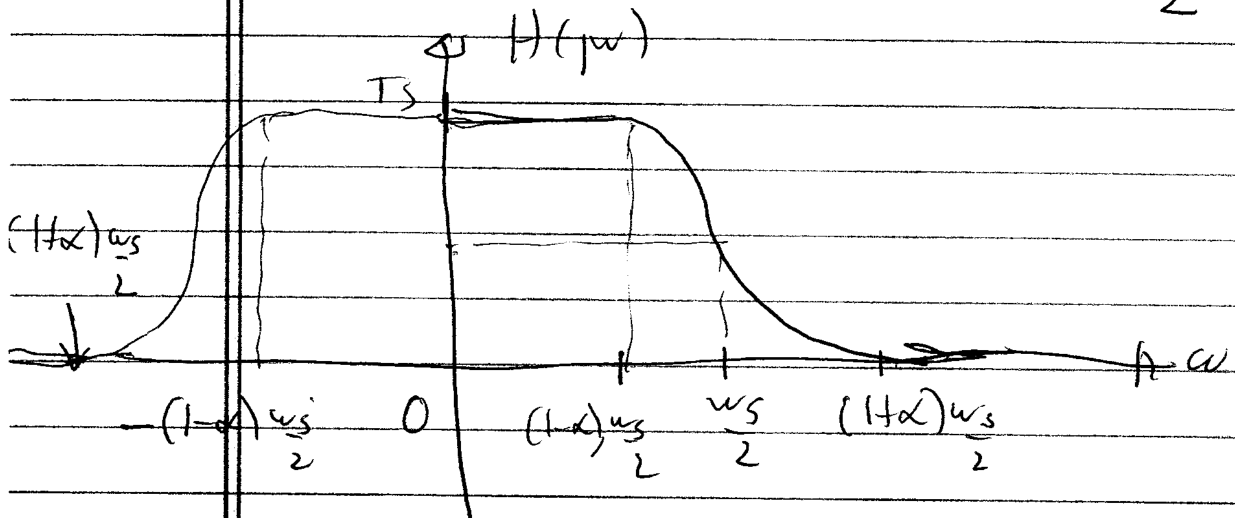
decays slowly like $1/t$, so in the sum

$$\sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s}$$

more terms need to be retained to evaluate $x(t)$ at same time t

A ~~more~~ practical choice would be to use the raised cosine ~~pulse~~ filter

$$H(j\omega) = \begin{cases} T_s & 0 \le |\omega| \le (1-\alpha)\frac{\omega_s}{2} \\ \frac{T_s}{2}\left(1 - \sin\left(\frac{T_s}{2\alpha}\left(|\omega| - \omega_s\right)\right)\right) & \text{for } (1-\alpha)\frac{\omega_s}{2} \le |\omega| \le (1+\alpha)\frac{\omega_s}{2} \\ 0 & |\omega| > (1+\alpha)\frac{\omega_s}{2} \end{cases} \qquad (19)$$



α ~~excess BW~~ controls the transition region

with impulse response

$$h(t) = \underbrace{\frac{\sin(\pi t/T)}{\pi t/T}}_{\text{ideal LP filter impulse response}} \frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2} \qquad (20)$$

Disadvantage

It requires

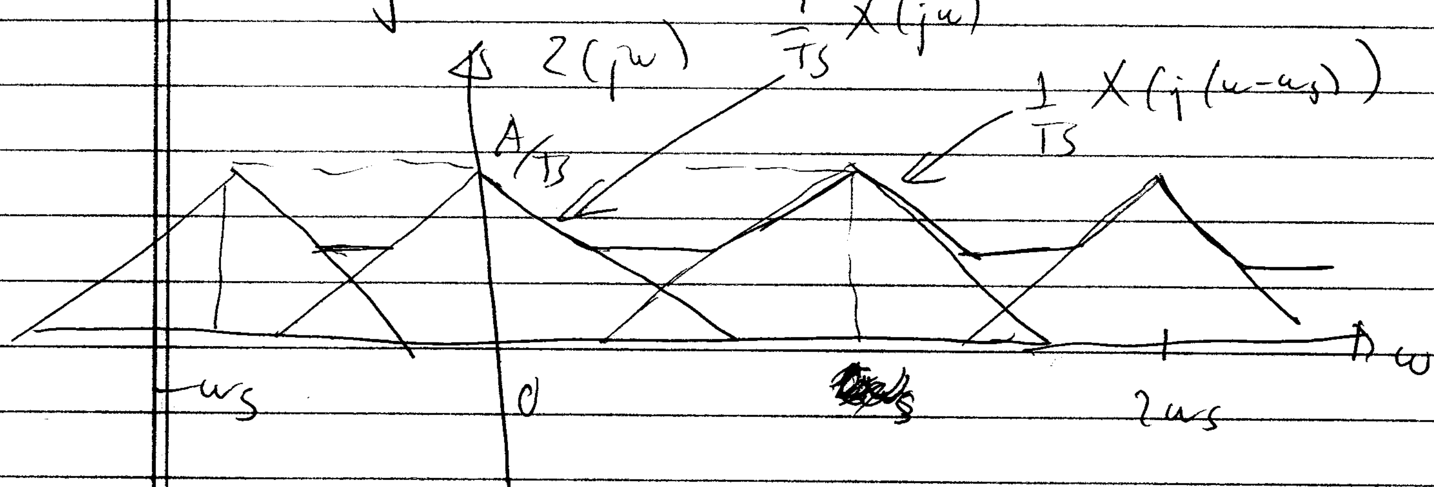
$$(1-\alpha)\frac{\omega_s}{2} > B$$

or equivalently

$\omega_s > (1-\alpha)\omega_s > 2B$ $\quad \alpha =$ ~~excess~~ ~~sam~~ oversampling ratio

Advantage $h(t)$ decays like $1/t^3$ fewer terms need to be used in interpolation formula (18)

Case 2: aliasing $\omega_s < 2B$



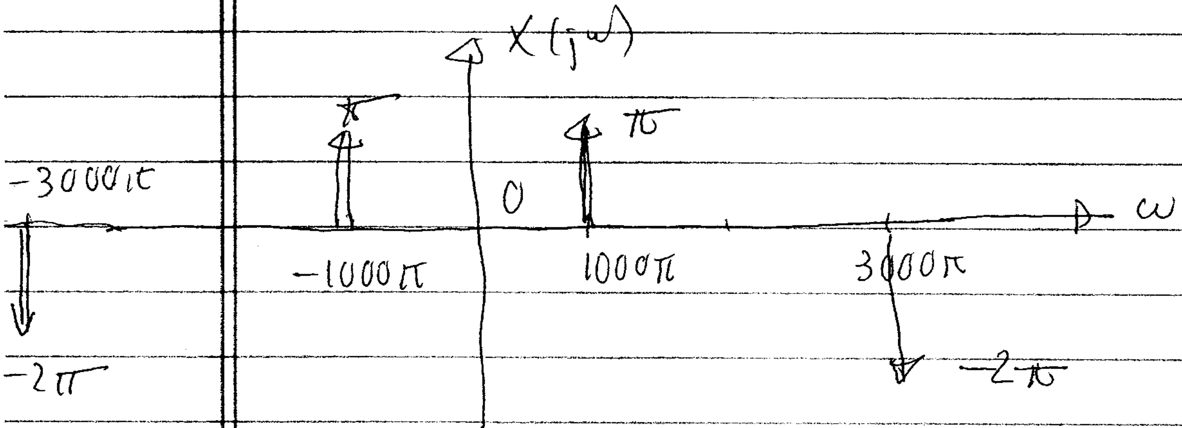
The copies $\frac{1}{T_S} X(j(\omega - k\omega_s))$ overlap and it is not possible to recover $X(j\omega)$ / $x(t)$ from $Z(j\omega)$ / $z(t)$

Example 1

$$x(t) = \cos(1000\pi t) - 2\cos(3000\pi t)$$

$\longleftrightarrow$

$$X(j\omega) = \pi\left[\delta(\omega - 1000\pi) + \delta(\omega + 1000\pi) - 2\left(\delta(\omega - 3000\pi) + \delta(\omega + 3000\pi)\right)\right]$$



Signal bandwidth = largest frequency

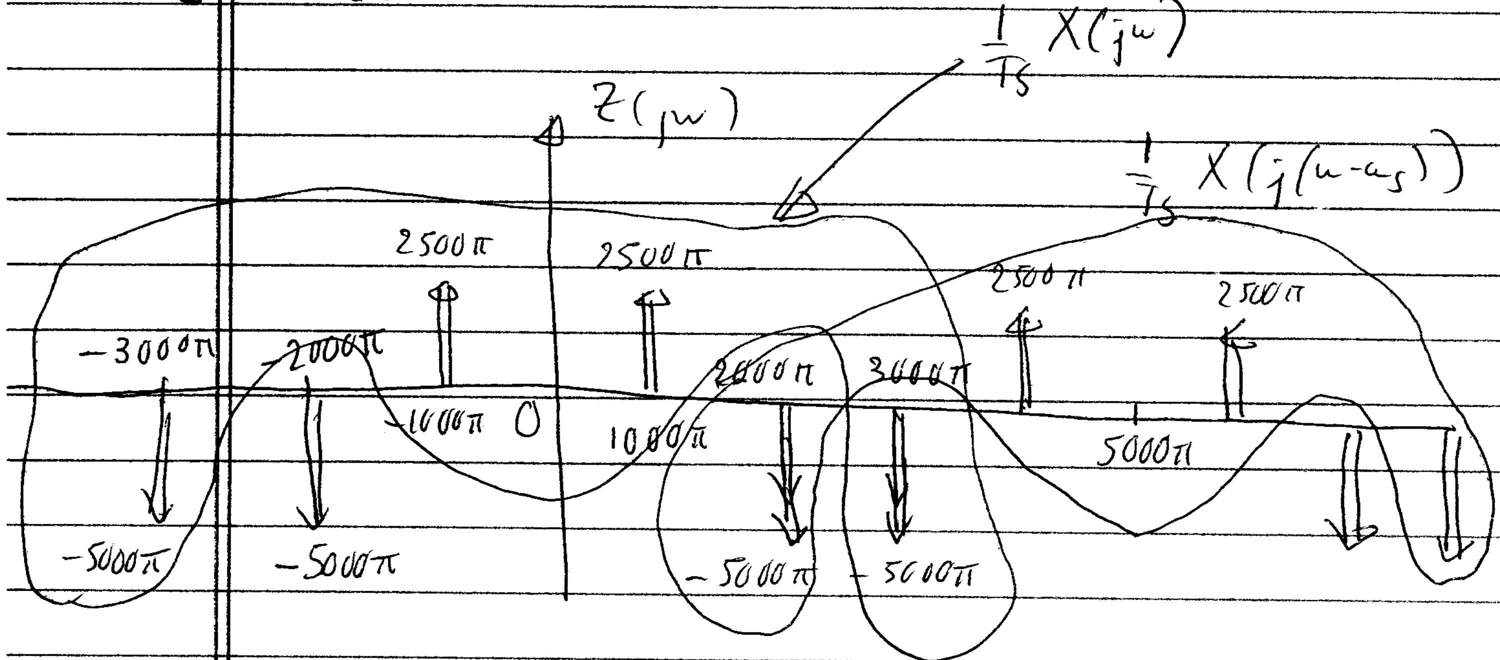
$B = 3000\pi$

so $\omega_N = 6000\pi$

= Nyquist frequency

Choice 1 : $\omega_s = 5000\pi < \omega_N$ does not satisfy the Nyquist sampling criterion

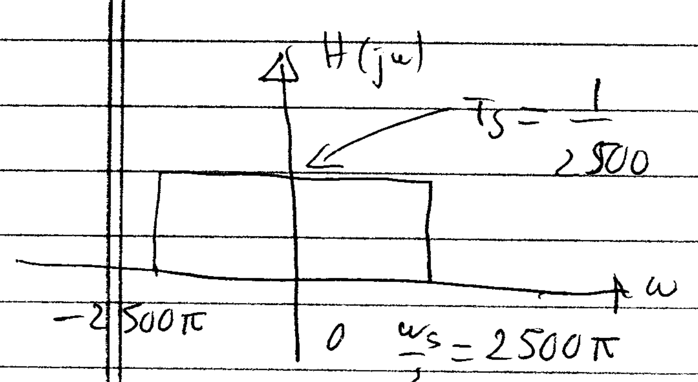
$$T_s = \frac{2\pi}{\omega_s} = \frac{1}{2500} = 0.4 \text{ msec}$$



Note that the frequency at 2000π is due to $\frac{1}{T_s} X(j(\omega-\omega_s))$

-2000π is due to $\frac{1}{T_s} X(j(\omega+\omega_s))$

Suppose now we use the ideal lowpass filter



for reconstruction

Then

$$\hat{X}(j\omega) = H(j\omega)\,Z(j\omega)$$

$$= \pi\big((\delta(\omega-1000\pi) + \delta(\omega+1000\pi))$$

$$-2(\delta(\omega-2000\pi) + \delta(\omega+2000\pi)\big)$$

or

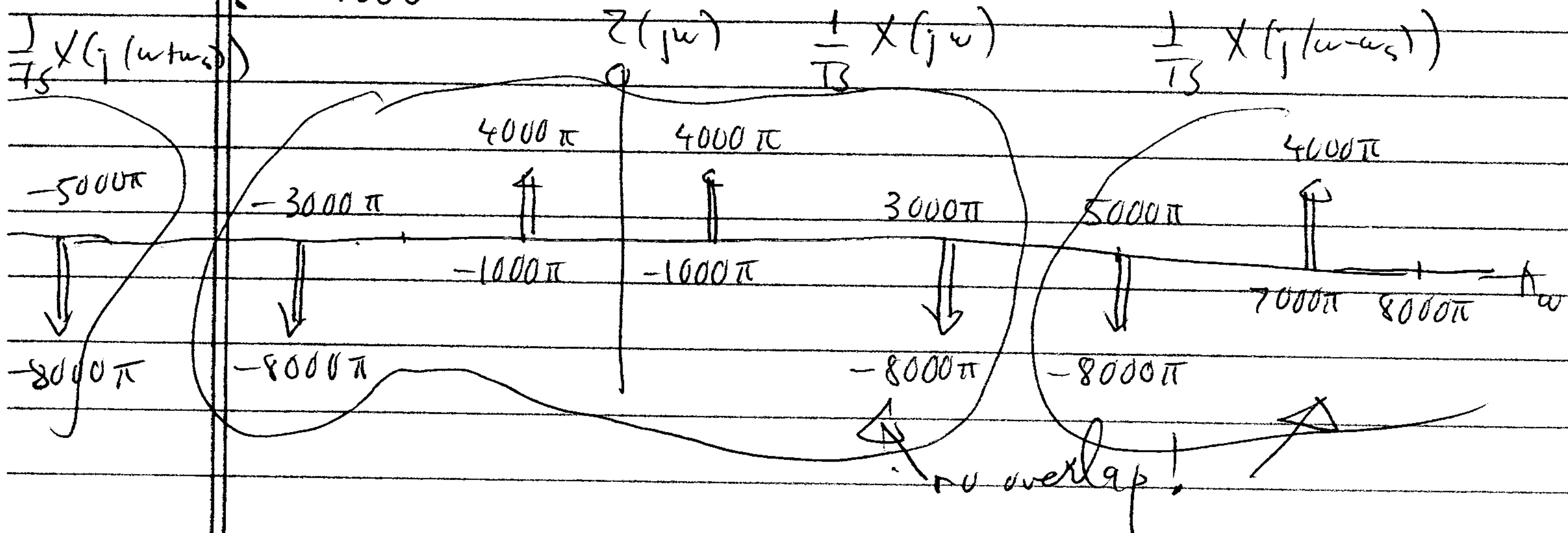
reconstructed correctly

$$\hat{x}(t) = \cos(1000\pi t) - 2\cos(2000\pi t)$$

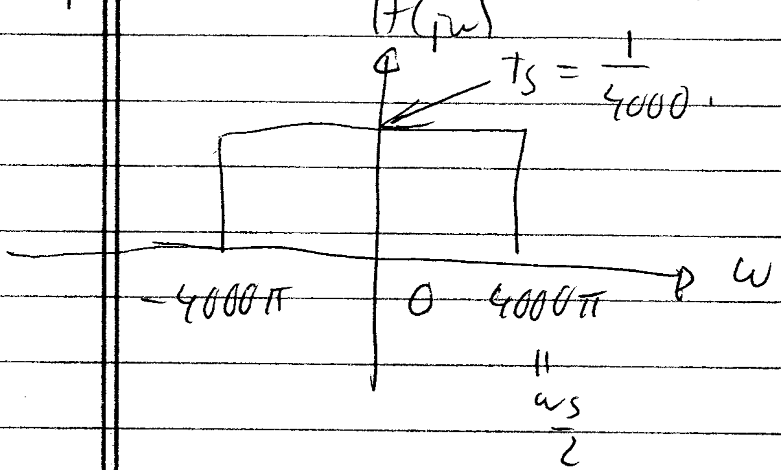
cosine at the <u>wrong</u> frequency!

<u>Choice 2</u>: $\omega_s = 8000\pi > \omega_N$ satisfies the Nyquist criterion

$$T_s = \frac{2\pi}{\omega_s} = \frac{1}{4000} = 0.25 \text{ msec}$$



and if we use the ideal low pass filter



we get

$$\hat{x}(t) = x(t) = \cos(1000\pi t) - 2\cos(3000\pi t)$$

exact reconstruction