Solution of the differential equation

$$y^{(2)}(t) + 5y^{(1)}(t) + 6y(t) = x^{(1)}(t) - 2x(t) \qquad (1)$$

with initial conditions

$$y(0_-) = 1, \quad y^{(1)}(0_-) = -1 \qquad (2)$$

and input $x(t) = u(t)$, $t \geq 0$

<u>Method 1</u>: ZIR + ZSR. We first evaluate the <u>ZIR</u>.

The characteristic polynomial is

$$a(s) = s^2 + 5s + 6 = (s+2)(s+3)$$

so

$$y_{ZIR}(t) = A_1 e^{-2t} + A_2 e^{-3t}$$

and

$$y^{(1)}_{ZIR}(t) = -2A_1 e^{-2t} - 3A_2 e^{-3t}.$$

Matching the initial conditions (2) gives

$$y_{ZIR}(0_-) = A_1 + A_2 = 1 \qquad (3)$$

$$y^{(1)}_{ZIR}(0_-) = -2A_1 - 3A_2 = -1 \qquad (4)$$

Performing the linear combination $2(3)+(4)$ gives

$$-A_2 = 1 \Rightarrow A_1 = 1 - A_2 = 2$$

so $y_{ZIR}(t) = 2e^{-2t} - e^{-3t}$ (5)

for $t \geqslant 0$.

To find the zero state response (ZSR)

$$y_{ZSR}(t) = \int_0^\infty h(t-\tau)\, x(\tau)\, d\tau \qquad (6)$$

we must first evaluate the impulse response $h(t)$.

Step 1: Solve

$$z^{(2)}(t) + 5z^{(1)}(t) + 6\,z(t) = x(t) = \delta(t)$$

with $z^{(1)}(0_-) = z(0_-) = 0$. Equivalently we solve

$$z^{(2)}(t) + 5z^{(1)}(t) + 6\,z(t) = 0 \qquad \text{for } t > 0 \qquad (7)$$

with $z^{(1)}(0_+) = 1$, $z(0_-) = 0$ [the impulse $\delta(t)$ switches the initial condition for $z^{(1)}(t)$ from 0 at $t = 0_-$ to 1 at $t = 0_+$, as explained in class].

Since the equation (7) is homogeneous we have

$$z(t) = B_1 e^{-2t} + B_2 e^{-3t}$$

for $t \geqslant 0$, so

$$z(0_+) = B_1 + B_2 = 0 \qquad (8)$$

$$z^{(1)}(0_+) = -2B_1 - 3B_2 = 1 \qquad (9)$$

Performing the linear combination $2\,(8)+(9)$ gives

$$-B_2 = 1 \quad \text{so} \quad B_1 = 1$$

and

$$z(t) = [e^{-2t} - e^{-3t}]\,u(t) \qquad (10)$$

<u>Step</u> 2: We have

$$h(t) = z^{(1)}(t) - 2\,z(t)$$

where

$$z^{(1)}(t) = [-2\,e^{-2t} + 3\,e^{-3t}]\,u(t) + [e^{-2t} - e^{-3t}]\,\delta(t)$$

$$= [-2\,e^{-2t} + 3\,e^{-3t}]\,u(t)$$

since $(e^{-2t} - e^{-3t})\big|_{t=0} = 0$. Thus

$$h(t) = (-4\,e^{-2t} + 5\,e^{-3t})\,u(t)$$

$$= \text{unit impulse response} \qquad (11)$$

The ZSR is then given by

$$y_{ZSR}(t) = \int_0^\infty h(t-\tau)\, x(\tau)\, d\tau$$

$$= \int_0^\infty \left[-4e^{-2(t-\tau)} + 5e^{-3(t-\tau)}\right] \underbrace{u(t-\tau)\,u(\tau)}_{=\begin{cases} 1 & \text{for } 0 \le t \le \tau \\ 0 & \text{otherwise}\end{cases}} d\tau$$

$$= \int_0^t \left[-4e^{-2(t-\tau)} + 5e^{-3(t-\tau)}\right] d\tau$$

$$\underset{v = t-\tau}{\overset{\downarrow}{=}} \int_0^t \left[-4e^{-2v} + 5e^{-3v}\right] dv$$

$$= 2\, e^{-2v}\Big|_0^t - \frac{5}{3}\, e^{-3v}\Big|_0^t$$

$$= 2\left(e^{-2t} - 1\right) - \frac{5}{3}\left(e^{-3t} - 1\right)$$

$$= \left[2e^{-2t} - \frac{5}{3}e^{-3t} - \frac{1}{3}\right] u(t) \qquad (12)$$

The complete solution is therefore

$$y(t) = y_{ZIR}(t) + y_{ZSR}(t)$$
$$= \left[4e^{-2t} - \frac{8}{3} e^{-3t} - \frac{1}{3} \right] u(t) \qquad (13)$$

Let's now try Method 2: Homogeneous plus particular solution.

Particular solution: Since $x(t) = u(t) = cst$ for $t \geq 0$ we can try $y_p(t) = C = cst$. We have

$$\underbrace{y_p^{(2)}(t)}_{=0} + 5 \underbrace{y_p^{(1)}(t)}_{=0} + 6 \underbrace{y_p(t)}_{C} = \underbrace{x^{(1)}(t)}_{=0} - 2 \underbrace{x(t)}_{=1}$$

for $t > 0$ so $C = -\frac{1}{3}$ and $y_p(t) = -\frac{1}{3}$ (14)

Homogeneous solution: We have

$$y_h(t) = D_1 e^{-2t} + D_2 e^{-3t} \qquad (15)$$

so

$$y_h(t) + y_p(t) = D_1 e^{-2t} + D_2 e^{-3t} - \frac{1}{3} \qquad (16)$$

All what is left is applying the initial conditions to the complete solution (16). However one tricky aspect is that the initial conditions that must be applied are not at $t=0_-$, but at $t=0_+$. We know that the initial conditions at 0_- are given by (2). But because $x(t)=u(t)$, the right hand side of (1) is

$$x^{(1)}(t)-2x(t)=\delta(t)-2u(t) \qquad (17)$$

so the right hand side hase an impulse! By using the same reasoning as for ~~the~~ finding the initial conditions for $z(t)$ at $t=0_+$, we find that the impulse in (17) switches the initial condition for $y^{(1)}(t)$ from $y^{(1)}(0_-)=-1$ to $y^{(1)}(0_+)=0$. (18) (increase by 1). The initial condition

$$y(0_+)=y(0_-)=1 \ . \qquad (19)$$

Now by applying initial conditions (18) and (19) to the complete solution (16) we find

$$D_1 + D_2 - \frac{1}{3} = 1 \qquad (20)$$

$$-2D_1 + 3D_2 = 0 \qquad (21)$$

Performing the linear combination $2\cdot(20) + (21)$ gives

$$-D_2 - \frac{2}{3} = 2 \quad \text{so} \quad D_2 = -\frac{8}{3}$$

and $D_1 = \frac{4}{3} - D_2 = 4$.

This gives

$$y_h(t) = \left(4e^{-2t} - \frac{8}{3}e^{-3t}\right)u(t) \neq y_{ZIR}(t)$$

$$y_p(t) = -\frac{1}{3}u(t) \neq y_{ZSR}(t)$$

and

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \left(4e^{-2t} - \frac{8}{3}e^{-3t} - \frac{1}{3}\right)u(t) \\ &= y_{ZIR}(t) + y_{ZSR}(t), \end{aligned}$$

as expected. Everything fits.