

EEC 250 Linear Systems and Signals

Lecture 11

Topics: a) DT and CT observability

b) Observability tests

c) Decomposition in observable/unobservable parts

DT observability: A state-space model

$$\underline{x}(k+1) = A \underline{x}(k) + B \underline{u}(k) \quad , \quad \underline{x}(0) = \underline{x}_0 \quad (1)$$

$$\underline{y}(k) = C \underline{x}(k) + D \underline{u}(k) \quad (2)$$

is said to be observable, if given the dynamics (A, B, C, D) , we can uniquely reconstruct the initial state \underline{x}_0 from the knowledge of the input $\underline{u}(k)$ and output $\underline{y}(k)$ for $k \geq 0$.

If we denote the matrix impulse response of the system (1)-(2) as

$$H(k) = D(k) \delta(k) + (A^{k-1} B \underline{1}(k-1))$$

where

$$\underline{1}(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is the unit step response, the output of the system (1)-(2) can be expressed as

$$\underline{y}(k) = ZIR + ZSR \quad (3)$$

where

$$ZSR = H(k) * u(k) = \sum_{l=0}^k H(k-l) u(l) \quad (4a)$$

is known, and

$$ZIR = CA^k \underline{x}_0. \quad (4b)$$

Since the ZSR is completely specified by the input and system dynamics, the problem of recovering the initial state \underline{x}_0 from the input and output reduces to one of finding \underline{x}_0 from the ZIR. In other words, we can assume without loss of generality that $u(k) \equiv 0$ in which case

$$y(k) = ZIR = CA^k \underline{x}_0. \quad (5)$$

The state \underline{x}_0 is said to be unobservable over the interval $[0, q-1]$ if $y(k) = 0$ for $0 \leq k \leq q-1$, so that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \underline{x}_0 = \underline{0}. \quad (6)$$

Thus, \underline{x}_0 is unobservable over $[0, q-1]$ if it is indistinguishable from the zero state $\underline{x}(0) = \underline{0}$ by observing the first q outputs of the system. Let

$$O_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \quad (7)$$

be the q -step observability matrix. Then x_0 is unobservable over $[0, q-1]$ if $x_0 \in N_q \triangleq N(O_q)$.

In the study of observability, the unobservable spaces play a role similar to that of the spaces R_q for reachability. The only difference is that R_q corresponds to the space of states which are reachable in q steps, whereas N_q is the space of states which are not observable in q steps.

Properties of N_q : 1) $N_{q+1} \subset N_q$. This property just states that for a given system, fewer states will be unobservable in $q+1$ steps than in q steps. Specifically if $y(k) = 0$ for $0 \leq k \leq q$, then $y(k) = 0$ for $0 \leq k \leq q$.

2) If $N_{q+1} = N_q$, then $N_s = N_q$ for all $s \geq q$. This property indicates that if the unobservable space N_q stops decreasing as we increase the interval of observation from $[0, q-1]$ to $[0, q]$, it will stop decreasing forever. To prove the above result, we only need to show that $N_{q+2} = N_{q+1} = N_q$. By induction, this will imply $N_s = N_q$ for $s \geq q$. To prove that $N_q = N_{q+2}$, it is only necessary to show that $N_q \subset N_{q+2}$, since we already know that $N_{q+2} \subset N_{q+1} = N_q$.

Let $x_0 \in N_q$, i.e.

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \underline{x}_0 = \underline{0} \quad (8a)$$

Since $\mathcal{N}_q = \mathcal{N}_{q+1}$, we must have $CA^q \underline{x}_0$. This implies

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} A \underline{x}_0 = \underline{0}, \quad (8b)$$

i.e. $A \underline{x}_0 \in \mathcal{N}_q$. But since $\mathcal{N}_q = \mathcal{N}_{q+1}$, we have also $A \underline{x}_0 \in \mathcal{N}_{q+1}$, and by combining this property with (8a) we find

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q+1} \end{bmatrix} \underline{x}_0 = \underline{0} \quad (9)$$

which shows $\underline{x}_0 \in \mathcal{N}_{q+2}$, as desired.

3) If A is an $n \times n$ matrix $\mathcal{N}_{n+1} = \mathcal{N}_n$. Combining this property with the first two, this implies that if a state \underline{x}_0 cannot be observed in n steps, it will never be observed.

This result is a consequence of the Cayley-Hamilton Theorem.

From

$$A^n = -a_1 A^{n-1} - a_2 A^{n-2} \dots - a_n I \quad (10)$$

we deduce

$$CA^n = -a_1 CA^{n-1} - a_2 CA^{n-2} \dots - a_n C \quad (11)$$

Then, if $\underline{x}_0 \in \mathcal{N}_n$, i.e. $CA^k \underline{x}_0 = 0$ for $0 \leq k \leq n-1$, we have also $CA^n \underline{x}_0 = 0$, so that $\underline{x}_0 \in \mathcal{N}_{n+1}$.

Definition: a) The unobservable space \mathcal{N} of the system (1)-(2) is the space constituted by the states \underline{x}_0 such that for $u(k) \equiv 0$ for $k \geq 0$, the outputs $\underline{y}(k) = 0$ for all $k \geq 0$. b) A system is said to be observable if it does not have any nonzero unobservable state, i.e. if $\mathcal{N} = \{ \underline{0} \}$.

From the properties 1)-3) of the unobservable spaces \mathcal{N}_q , we obtain the following result.

Theorem 1: a) The unobservable space $\mathcal{N} = \mathcal{N}_n$, the right null space of the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (12)$$

b) The system (1)-(2) is observable if and only if O_n has rank n .

Note that since the observability matrix O_n has size $p \times n$, i.e.

it has more rows than columns, O_n will have rank n if and only if its n columns are linearly independent.

Example: The system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1/6 \\ 1/3 & 1/2 \end{bmatrix}}_A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

is not observable since the observability matrix

$$O_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/3 & 1/3 \end{bmatrix}$$

has only rank 1. The unobservable space is given by

$$\mathcal{N} = \mathcal{N}(O_2) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

CT observability: Consider the system

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t), \quad \underline{x}(0) = \underline{x}_0 \quad (13a)$$

$$\underline{y}(t) = C \underline{x}(t) + D \underline{u}(t) \quad (13b)$$

Given the dynamics (A, B, C, D) , the observability problem consists in recovering the initial state \underline{x}_0 from $\underline{u}(t)$ and $\underline{y}(t)$ for $0 \leq t \leq T$. As in

the DT case, the ZSR is completely determined by the input and dynamics, so that the given problem reduces to finding \underline{x}_0 from

$$\underline{y}(t) - \text{ZSR} = \text{ZIR} = C e^{At} \underline{x}_0 \quad (14)$$

In other words, we can assume without loss of generality that $u(t) \equiv 0$ in which case

$$\underline{y}(t) = C e^{At} \underline{x}_0 \quad (15)$$

Since the output $\underline{y}(t)$ given by (15) admits an infinite number of derivatives, for ϵ sufficiently small, $\underline{y}(t)$ can be expanded in power series as

$$\underline{y}(t) = \sum_{k=0}^{\infty} \underline{y}^{(k)}(0) \frac{t^k}{k!} \quad (16)$$

for $0 \leq t \leq \epsilon$, from which we see that the knowledge of $\underline{y}(t)$ for $0 \leq t \leq \epsilon$ is equivalent to the knowledge of the derivatives $\{\underline{y}^{(k)}(0), k \geq 0\}$. As in our study of CT reachability, we shall distinguish between instantaneous observability, where the goal is to reconstruct \underline{x}_0 from the derivatives $\underline{y}^{(k)}(0)$ for $k \geq 0$, and finite-time observability, where \underline{x}_0 is recovered from $\underline{y}(t)$ with $0 \leq t \leq T$.

The above discussion about the equivalence of $\underline{y}(t)$ for $0 \leq t \leq \epsilon$ and $\underline{y}^{(k)}(0)$ for $k \geq 0$ suggests that these two concepts are equivalent, as will be verified below.

Instantaneous observability: Obviously, it is impractical to try to reconstruct \underline{x}_0 from $y^{(k)}(0)$ with $k \geq 0$, since this assumes the output $y(t)$ is noiseless and can be differentiated an arbitrary number of times. In practice $y(t)$ contains a noise component which is amplified after differentiation, so that successive derivatives are dominated by noise. Nevertheless, the problem of reconstructing \underline{x}_0 from $y^{(k)}(0)$ with $k \geq 0$ provides a simple link between CT and DT observability. This is due to the fact that

$$\underline{y}^{(k)}(0) = CA^k \underline{x}_0. \quad (17)$$

The state \underline{x}_0 is said to be unobservable from the first $q-1$ derivatives of $y(t)$ at $t=0$, if $y^{(k)}(0) = 0$ for $0 \leq k \leq q-1$, so that \underline{x}_0 belongs to the right null space \mathcal{N}_q of

$$O_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}.$$

The unobservable space \mathcal{N}_q is thus identical to the space of states that cannot be observed from the first q observations of a DT system with the same (C, A) pair.

From the DT results, we can conclude that the space \mathcal{W} of states \underline{x}_0 such that $y^{(k)}(0) = 0$ for all $k \geq 0$ coincides with \mathcal{N}_n .

Consequently, the system is instantaneously observable provided that $N_n = \{0\}$, i.e. if and only if the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n .

Finite-time observability: In this case, we are given

$$y(t) = ZIR = C e^{At} x_0 \quad (18)$$

for $0 \leq t \leq T$ and we seek to recover x_0 . The state x_0 is said to be unobservable over $[0, T]$ if the corresponding output $y(t) \equiv 0$ for $0 \leq t \leq T$.

To determine whether this occurs, we can compute the output energy

$$E(T) = \frac{1}{2} \int_0^T \|y(t)\|^2 dt = \frac{1}{2} \int_0^T y^T(t) y(t) dt = \frac{1}{2} x_0^T M(T) x_0 \quad (19a)$$

where

$$M(T) = \int_0^T e^{A^T t} C^T C e^{At} dt \quad (19b)$$

is the observability Gramian of the system over $[0, T]$. Then $y(t) \equiv 0$ for $0 \leq t \leq T$ if and only if the energy $E(T) = 0$, or equivalently if the initial state x_0 satisfies

$$x_0^T M(T) x_0 = 0. \quad (20)$$

Since the matrix $M(T)$ is symmetric nonnegative definite, the quadratic form $\underline{x}_0^T M(T) \underline{x}_0$ is zero if and only if the vector \underline{x}_0 belongs to the right null space of $M(T)$, i.e.

$$M(T) \underline{x}_0 = 0. \quad (21)$$

Thus, the space $\mathcal{N}(T)$ of states which are unobservable over $[0, T]$ is formed by the right null space of the observability Gramian $M(T)$. As usual, we say that the system (13a)-(13b) is observable over $[0, T]$ if $\mathcal{N}(T) = \{0\}$. From the above characterization of $\mathcal{N}(T)$ we obtain the following observability test.

Theorem 2: The system (13a)-(13b) is observable over $[0, T]$ if and only if the observability Gramian $M(T)$ is positive definite.

To verify that when $M(T)$ is invertible, the state \underline{x}_0 can be reconstructed from the output $y(t)$ for $0 \leq t \leq T$, we only need to observe that

$$\int_0^T e^{A^T t} C^T y(t) dt = M(T) \underline{x}_0 \quad (22)$$

so that

$$\underline{x}_0 = M^{-1}(T) \int_0^T e^{A^T t} C^T y(t) dt. \quad (23)$$

The observability Gramian provides a quantitative measure of the system observability. This is due to the fact that the energy of the output over $[0, T]$ generated by the initial state \underline{x} can be expressed as

$$E(T) = \underline{x}^T M(T) \underline{x} . \quad (24)$$

Obviously, if the initial state is scaled by a factor a , the output energy $E(T)$ is scaled by a^2 . Dividing $E(T)$ by the squared length of the initial state \underline{x} to remove this scale factor, we obtain the normalized energy

$$\bar{E}(T) = \frac{\underline{x}^T M(T) \underline{x}}{\underline{x}^T \underline{x}} \quad (25)$$

which measures the degree of observability of each direction in the state-space. Let λ_{\min} be the smallest eigenvalue of $M(T)$, and let \underline{p}_{\min} be the corresponding eigenvector, where we assume that \underline{p}_{\min} has been normalized so that $\underline{p}_{\min}^T \underline{p}_{\min} = 1$. Then the normalized output energy $\bar{E}(T)$ is minimized for $\underline{x} = \underline{p}_{\min}$, with

$$\bar{E}(T)_{\min} = \lambda_{\min} . \quad (26)$$

If λ_{\min} is small, in particular with respect to any noise component that might be present in the output, it will be almost impossible to observe the initial state \underline{p}_{\min} . On the other hand, if the SNR

$\frac{\lambda_{\min}}{N_0} \gg 1$, where N_0 is the amplitude of the noise component at the output of the matched filter

$$\int_0^T e^{A^T t} C^T y(t) dt = \int_0^T G(T-t) y(t) dt \quad (27)$$

with $G(t) = e^{A^T (T-t)} C^T$, the system will be easy to observe.

Equivalence of instantaneous and finite-time CT observability: To prove the equivalence of these two observability concepts, we only need to show that the spaces \mathcal{N} and $\mathcal{N}(T)$ of instantaneous and finite-time unobservable states are the same. This is accomplished by proving the following result.

Theorem 3: The right null spaces \mathcal{N} and $\mathcal{N}(T)$ of the matrices

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and $M(T)$ are the same.

Proof: The argument is identical to the one employed in Lecture 10 to show that the left null spaces of $R_n = [B \ AB \ \dots \ A^{n-1}B]$ and $W(T)$ are the same. Specifically, if \underline{x} is a vector in the right null space of $M(T)$, we

have

$$\begin{aligned} 0 = \underline{x}^T M(T) \underline{x} &= \int_0^T \underline{x}^T e^{A^T t} C^T C e^{A t} \underline{x} dt \\ &= \int_0^T \|\underline{y}(t)\|^2 dt \end{aligned} \quad (28)$$

with $\underline{y}(t) = C e^{A t} \underline{x}$. This implies

$$\underline{y}(t) = C e^{A t} \underline{x} = 0 \quad \text{for } 0 \leq t \leq T. \quad (29)$$

Differentiating (29) k times, and setting $t=0$ gives

$$\underline{y}^{(k)}(0) = C A^k \underline{x} = 0 \quad \text{for all } k \geq 0, \quad (30)$$

so that \underline{x} belongs to the right null space \mathcal{N} of σ_n .

Conversely, let \underline{x} be in the right null space of σ_n . By observing that

$$e^{A t} = \alpha_1(t) A^{n-1} + \alpha_2(t) A^{n-2} + \dots + \alpha_n(t) I_n \quad (31)$$

we see that

$$\underline{y}(t) = C e^{A t} \underline{x} = \sum_{i=1}^n \alpha_i(t) C A^{n-i} \underline{x} \equiv 0 \quad (32)$$

which in turn implies that \underline{x} is in the right null space $\mathcal{N}(T)$ of $M(T)$.

Thus, all concepts of observability for DT and CT systems with the same (C, A) pair lead to the same tests.

Geometric characterization of the unobservable space \mathcal{N} : The right null space \mathcal{N} of the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has the following properties

- 1) $\mathcal{N} \subset \mathcal{N}(C)$ = the right null space of C . This is just due to the fact that the first block row of O_n is C , so that $O_n \underline{x} = \underline{0}$ implies $C\underline{x} = \underline{0}$
- 2) \mathcal{N} is A -invariant, i.e. $A\mathcal{N} \subset \mathcal{N}$. To see this, consider the discrete-time system (1)-(2), and assume that \underline{x}_0 is an unobservable state, so that for $\underline{x}(0) = \underline{x}_0$, we have $y(k) = 0$ for all $k \geq 0$. Then $\underline{x}(1) = A\underline{x}_0$ can be viewed as a new initial state, for which the corresponding output $y(k) = 0$ for $k \geq 1$. This shows that if \underline{x}_0 is unobservable, so is $A\underline{x}_0$, so that \mathcal{N} is A -invariant.

In fact, it can be shown that \mathcal{N} is the largest A invariant space contained in $\mathcal{N}(C)$.

System decomposition in observable/unobservable parts: The geometric characterization of \mathcal{N} can be used to decompose the given system in its observable and unobservable parts. We assume that \mathcal{N} has dimension

$n-s$, and let $\{t_{s+1} \dots t_n\}$ be a basis of \mathcal{N} . This basis can be completed into a basis of \mathbb{R}^n by selecting s additional vectors $\{t_1 \dots t_s\}$ such that the columns of

$$T = [t_1 \dots t_s \ t_{s+1} \dots t_n]$$

are linearly independent, so that T is invertible.

Since \mathcal{N} is contained in the right null space of C , we have

$$C \underbrace{[t_1 \dots t_s \ t_{s+1} \dots t_n]}_T = \underbrace{[C_0 \mid 0]}_{\substack{s \quad n-s}} \quad (33)$$

Similarly, the A -invariance property of \mathcal{N} implies

$$A [t_1 \dots t_s \ t_{s+1} \dots t_n] = [t_1 \dots t_s \ t_{s+1} \dots t_n] \begin{bmatrix} A_0 & 0 \\ A_{0\bar{0}} & A_{\bar{0}} \end{bmatrix} \quad (34)$$

$\begin{array}{c} \uparrow \delta \\ \downarrow n-s \end{array}$

for some matrices $A_0, A_{\bar{0}}$ and $A_{0\bar{0}}$.

By applying the similarity transformation $\underline{x}(k) = T \tilde{x}(k)$, the system (A, B, C, D) is thus transformed into

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_0 & 0 \\ A_{0\bar{0}} & A_{\bar{0}} \end{bmatrix} \quad \tilde{B} = T^{-1}B = \begin{bmatrix} B_0 \\ \dots \\ B_{\bar{0}} \end{bmatrix} \quad (35)$$

$\begin{array}{c} \uparrow \delta \\ \downarrow n-s \end{array}$

$$\tilde{C} = CT = [C_0 \ 0] \quad \tilde{D} = D.$$

The transformed system has the following properties.

1) The pair (C_0, A_0) is observable. To prove this, note that since the null space \mathcal{N} of the observability matrix O_n has dimension $n-s$, the dimension of the column space of O_n , i.e. the rank of O_n , must be s . Then

$$\begin{aligned} \tilde{O}_n &= \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C_0 & | & 0 \\ C_0 A_0 & | & 0 \\ \vdots & | & \vdots \\ C_0 A_0^{n-1} & | & 0 \end{bmatrix} \\ &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = O_n T \end{aligned} \quad (36)$$

has also rank s , since the column space of O_n is unchanged by multiplying O_n on the right by the invertible matrix T . This implies that the rank of the matrix

$$\begin{bmatrix} C_0 \\ C_0 A_0 \\ \vdots \\ C_0 A_0^{n-1} \end{bmatrix}$$

is s , but since A_0 has size $s \times s$, the row space of this matrix is identical to that of

$$O(C_0, A_0) = \begin{bmatrix} C_0 \\ C_0 A_0 \\ \vdots \\ C_0 A_0^{s-1} \end{bmatrix}.$$

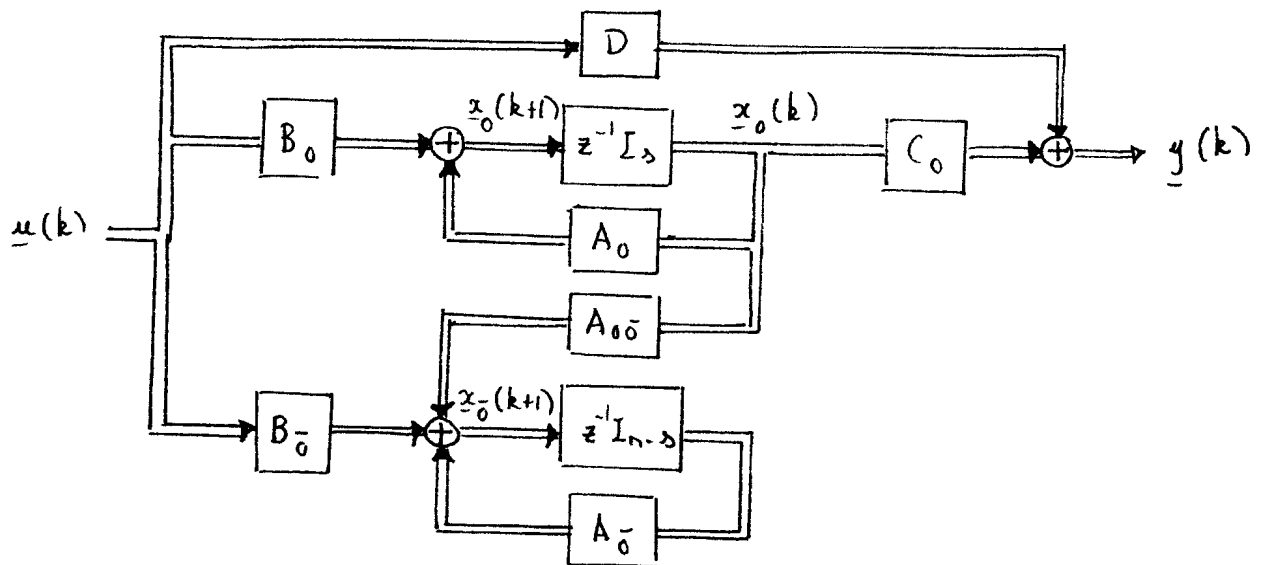
This implies that $\text{rank } O(C_0, A_0) = n$, so that (C_0, A_0) is observable.

2) The block diagrams of the transformed system

$$\begin{bmatrix} \underline{x}_0(k+1) \\ \underline{x}_0^-(k+1) \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ A_{00^-} & A_0^- \end{bmatrix} \begin{bmatrix} \underline{x}_0(k) \\ \underline{x}_0^-(k) \end{bmatrix} + \begin{bmatrix} B_0 \\ B_0^- \end{bmatrix} \underline{u}(k) \quad (37a)$$

$$\underline{y}(k) = [C_0 \quad 0] \begin{bmatrix} \underline{x}_0(k) \\ \underline{x}_0^-(k) \end{bmatrix} + D \underline{u}(k) \quad (37b)$$

takes the form



from which we see that the component $\underline{x}_0^-(k)$ of the transformed state vector $\tilde{\underline{x}}(k) = \begin{bmatrix} \underline{x}_0(k) \\ \underline{x}_0^-(k) \end{bmatrix}$ is unobservable, since it is completely disconnected

from the output $\underline{y}(k)$, which makes it impossible to reconstruct $\underline{x}_0^-(0)$ from the output.

3) The transfer function

$$\begin{aligned}
 H(z) &= C(zI - A)^{-1}B + D = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + D \\
 &= C_0(zI - A_0)^{-1}B_0 + D, \quad (38)
 \end{aligned}$$

where we have used the fact that

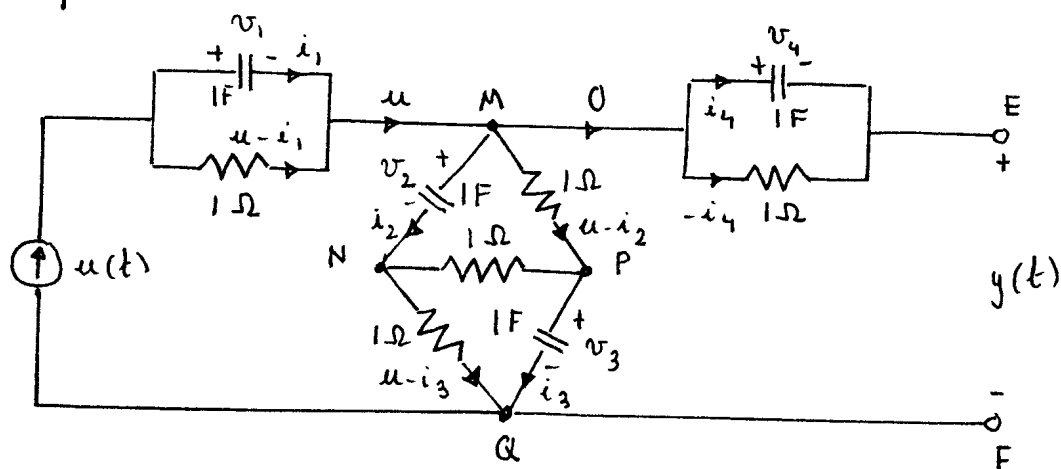
$$(zI - \tilde{A})^{-1} = \begin{bmatrix} (zI - A_0)^{-1} & 0 \\ (zI - A_0)^{-1}A_0\bar{o}(zI - A_0)^{-1} & (zI - A_0)^{-1} \end{bmatrix}$$

and

$$\tilde{C}(zI - \tilde{A})^{-1} = [C_0(zI - A_0)^{-1} \quad 0].$$

The identity (38) shows that the realization (A, B, C, D) of the transfer function $H(z)$ is nonminimal, since it requires n states, whereas the description (A_0, B_0, C_0, D) requires only $s < n$ states. This last realization may still be nonminimal if the pair (A_0, B_0) is not reachable

Example: Consider the circuit



By inspection, we see that

$$v_1 = u - i, \quad v_4 = -i_4$$

so that

$$i_1 = \dot{v}_1 = u - v_1, \quad i_4 = \dot{v}_4 = -v_4.$$

Writing the KVL along loops MPN and PNQ gives respectively

$$v_2 = u - i_2 + u - i_2 - i_3 = 2u - 2i_2 - i_3$$

$$v_3 = u - i_2 - i_3 + u - i_3 = 2u - i_2 - 2i_3,$$

which after solving for i_2 and i_3 yields

$$\begin{bmatrix} \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2u - v_2 \\ 2u - v_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2u - 2v_2 + v_3 \\ 2u + v_2 - 2v_3 \end{bmatrix}.$$

The state-space dynamics for the given circuit can therefore be expressed as

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 2/3 \\ 2/3 \\ 0 \end{bmatrix}}_B u(t).$$

To express the output $y(t)$ in terms of the states, we use KVL along the loop EMNQF. This yields

$$y = -v_4 + v_2 + u - i_3 = \frac{2}{3}(v_2 + v_3) - v_4 + \frac{u}{3},$$

which can be expressed in matrix form as

$$y(t) = \underbrace{\begin{bmatrix} 0 & \frac{2}{3} & \frac{2}{3} & -1 \end{bmatrix}}_C \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \frac{1}{3} u(t).$$

The observability matrix for this circuit is given by

$$O_4 = \begin{bmatrix} 0 & \frac{2}{3} & \frac{2}{3} & -1 \\ 0 & -\frac{2}{3^2} & -\frac{2}{3^2} & 1 \\ 0 & \frac{2}{3^3} & \frac{2}{3^3} & -1 \\ 0 & -\frac{2}{3^4} & -\frac{2}{3^4} & 1 \end{bmatrix}.$$

Its null space \mathcal{N} is spanned by the vectors

$$\underline{t}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{t}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

The vectors $\{\underline{t}_3, \underline{t}_4\}$ can be completed into a basis of \mathbb{R}^4 by selecting the vectors

$$\underline{t}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{t}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so that

$$T = [\underline{t}_1, \underline{t}_2, \underline{t}_3, \underline{t}_4] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}.$$

The transformed state variables and dynamics are

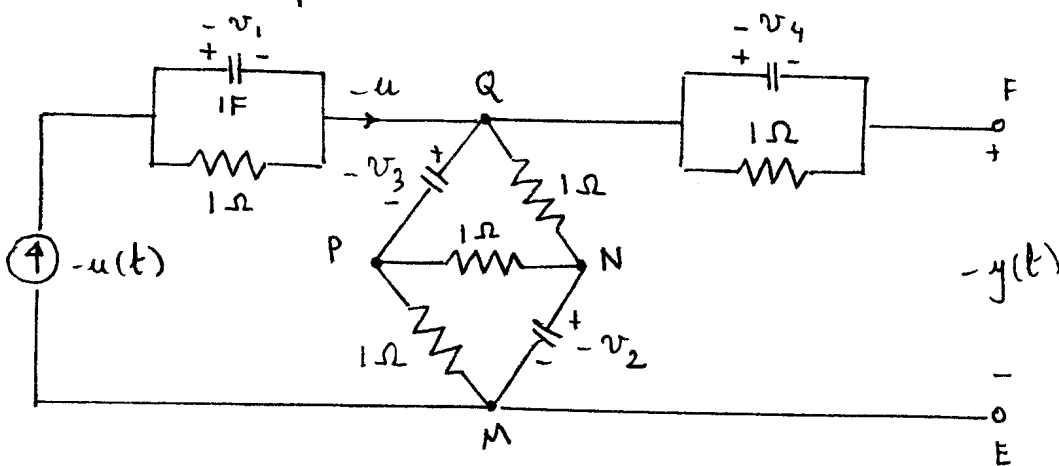
$$\tilde{\underline{x}} = T^{-1} \underline{x} = \left[\begin{array}{c} (v_2 + v_3)/2 \\ v_4 \\ \hline v_1 \\ (v_2 - v_3)/2 \end{array} \right] \begin{array}{l} \underline{x}_0 \\ \\ \underline{x}_0 \end{array}$$

and

$$\tilde{\underline{A}} = T^{-1} \underline{A} T = \left[\begin{array}{cc|cc} -1/3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right] \quad \tilde{\underline{B}} = T^{-1} \underline{B} = \left[\begin{array}{c} 2/3 \\ 0 \\ 1 \\ 0 \end{array} \right]$$

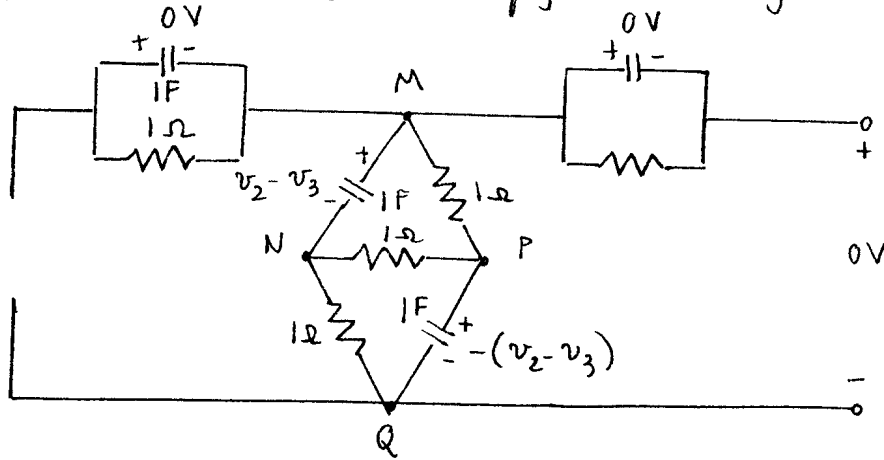
$$\tilde{\underline{C}} = \underline{C} T = \left[\begin{array}{cc|cc} 4/3 & -1 & 0 & 0 \end{array} \right] .$$

To obtain a physical interpretation of the lack of observability of the states $v_2 - v_3$ and v_1 , note that the given circuit can be redrawn by turning it upside down, and moving the blocks containing capacitors v_1 and v_4 as follows

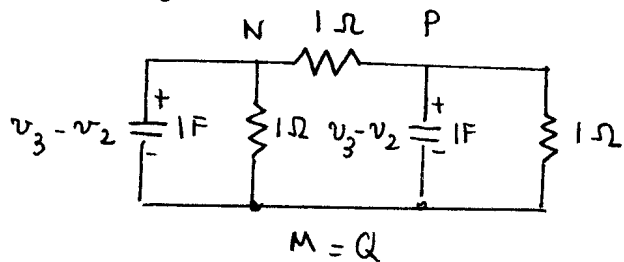


But this new circuit is identical to the original one, so that by applying the principle of superposition, we can either add or subtract their voltage

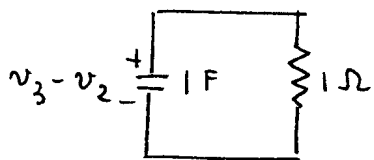
configurations. Adding the configurations yields



Replacing all branches with a 0V voltage drop by a short circuit yields

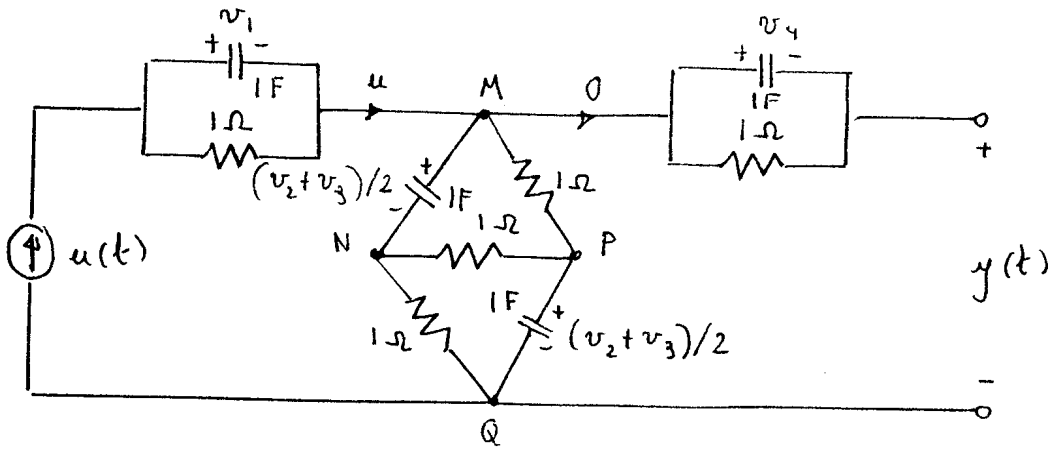


Since the voltages of nodes N and P are the same, no current flows through the 1Ω resistor linking nodes N and P, so that this 1Ω resistor can be removed. This gives the circuit

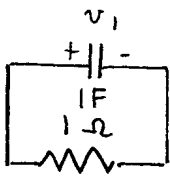


which satisfies the equation $\frac{d}{dt}(v_2 - v_3) = -(v_2 - v_3)$, as was obtained earlier.

On the other hand, subtracting the two voltage configurations gives

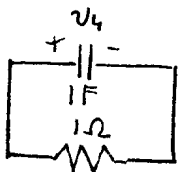


Note that this circuit involves only the variables v_1 , $(v_2 + v_3)/2$ and v_4 , so that the variable $(v_2 - v_3)/2$ is both unreachable and unobservable, since it is not affected by the input $u(t)$ and does not affect the output $y(t)$. To see that $v_1(t)$ is unobservable, note that when discussing observability, we can assume that the input $u(t) = 0$. In this case, no current flows through the branch arriving at M from the left, so that this branch can be cut, which yields the isolated subcircuit shown



below. This implies that v_1 is unobservable from the output $y(t)$.

Similarly, we note that the branch leaving M going to the right does not carry any current, so that it can be cut, thus yielding the isolated subcircuit shown below. This shows that v_4 is unaffected by



the input $u(t)$, so that it is unreachable.

Reachability / observability duality: Since the reachability and observability tests obtained earlier require respectively that the matrices

$$R_n = [B \ AB \ \dots \ A^{n-1}B]$$

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

should have rank n , we can conclude that the pair (C, A) is observable if and only if (A^T, C^T) is reachable. This implies the following eigenvalue / eigenvector observability test.

Theorem 4: The pair (C, A) is observable if and only if

$$\text{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}. \quad (39)$$

Since the matrix $sI - A$ has full rank for all values of s which are not eigenvalues of A , we need only to check whether

$$M(\lambda) = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

has rank n for all eigenvalues λ of A . Furthermore if there exists an eigenvalue λ for which $M(\lambda)$ does not have full rank, we can find a vector $p \neq 0$ such that

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \underline{p} = \underline{0}, \quad (40)$$

i.e. there exists a right eigenvector \underline{p} of A which is orthogonal to the rows of C . Physically, this means that the system is not observable whenever a natural mode $\underline{x}(t) = \underline{p} e^{\lambda t}$ of the system gives rise to an identically zero output, i.e. $y(t) = C \underline{p} e^{\lambda t} = 0$. In other words, a system is observable if all its natural modes are detectable from the output.

Remark: From the duality between reachability and observability, by observing that

$$A_o = A_c^T \quad b_o = c_c^T \quad c_o = b_c^T \quad (41a)$$

$$A_{ob} = A_{c_o}^T \quad b_{ob} = c_{c_o}^T \quad c_{ob} = b_{c_o}^T \quad (41b)$$

and noting that the controller and controllability realizations are reachable, we can immediately deduce the observer and observability realizations are observable. Furthermore, since the observer form is reachable if and only if in the transfer function

$$H(s) = \frac{b(s)}{a(s)},$$

the polynomials $a(s)$ and $b(s)$ are coprime, the controller form is observable if and only if $a(s)$ and $b(s)$ are coprime.