EEC 250 Linear Systems and Signals

Lecture 12

Topics: a) Kalman decomposition of a linear system

b) Minimal realizations

c) Similarity of minimal realizations

Kalman decomposition: In Lectures 9 and 11, it was shown how to decompose a state-space model

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{2}$$

into its reachable and unreachable parts, and its observable and unobservable parts, respectively. We now combine these two decompositions in order to obtain a four-part decomposition of the system into reachable and observable, reachable and unobservable, unreachable and observable, and unreachable and unobservable parts. If

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad R = [\,B\ AB\ \ldots\ A^{n-1}B\,]$$

denote respectively the observability and reachability matrices of the system, the four-part Kalman decomposition relies on the two spaces

$\mathcal{R}$ = reachable space = column space of R

$\mathcal{N}$ = unobservable space = right null space of 0

Note thate these two spaces are not uniquely defined, by opposition with their complements $\bar{\mathcal{R}}$ and $\bar{\mathcal{N}}$ which satisfy

$$\mathbb{R}^n = \bar{\mathcal{R}} \oplus \mathcal{R} \qquad \mathbb{R}^n = \bar{\mathcal{N}} \oplus \mathcal{N} \tag{3}$$

The spaces $\bar{\mathcal{R}}$ and $\bar{\mathcal{N}}$ are often called the unreachable and observable spaces, respectively, but this is a misnomer since the direct complement of a space inside $\mathbb{R}^n$ is not uniquely specified.

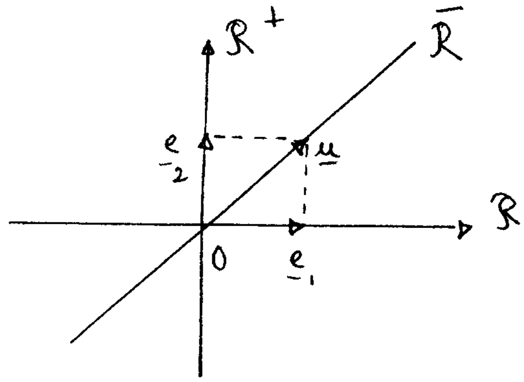
<u>Example</u>: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, the column space $\mathcal{R}$ of

$$R = [\,b \;\; A\,b\,] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is spanned by $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that $\mathcal{R}$ coincides with the horizontal axis of $\mathbb{R}^2$. Then any vector which is not colinear with $\underline{e}_1$ will span a complement $\bar{\mathcal{R}}$ of $\mathcal{R}$. Two possible choices are given by $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, in which case the space $\bar{\mathcal{R}}$ spanned by $\underline{u}$ is an oblique complement of $\mathcal{R}$, and $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which spans the orthogonal complement $\mathcal{R}^\perp$ of $\mathcal{R}$.



Since $\mathcal{R}$ and $\mathcal{N}$ are uniquely defined, they are the two spaces we shall use to obtain the four-part Kalman decomposition of the system. The decomposition proceeds as follows.

<u>Step 1</u>: Consider the intersection $V_{r\bar{o}} = \mathcal{R} \cap \mathcal{N}$ of $\mathcal{R}$ and $\mathcal{N}$. Vectors belonging to this space are reachable and unobservable. We can select a basis of $V_{r\bar{o}}$ and let $T_{r\bar{o}}$ be the matrix whose columns are the basis vectors.

<u>Step 2</u>: Construct complements of $\mathcal{R} \cap \mathcal{N}$ inside $\mathcal{R}$ and $\mathcal{N}$, respectively. These two spaces are denoted V_{ro} and $V_{\bar{r}\bar{o}}$ and satisfy

$$\mathcal{R} = V_{ro} \oplus (\mathcal{R} \cap \mathcal{N}) = V_{ro} \oplus V_{r\bar{o}} \tag{4}$$

$$\mathcal{N} = V_{\bar{r}\bar{o}} \oplus (\mathcal{R} \cap \mathcal{N}) = V_{\bar{r}\bar{o}} \oplus V_{r\bar{o}} \tag{5}$$

Note that V_{ro} and $V_{\bar{r}\bar{o}}$ are not uniquely specified, since as was observed earlier, the complement of a space inside another is not uniquely defined.

V_{ro} is the space of reachable and observable states. Selecting a basis for this space then yields a matrix T_{ro} whose columns are the basis vectors. Similarly, $V_{\bar{r}\bar{o}}$ is the space of unreachable and unobservable states. Again, we can choose a basis for $V_{\bar{r}\bar{o}}$ and let $T_{\bar{r}\bar{o}}$ be the matrix whose columns are the basis elements.

<u>Step 3</u>: We find a complement $V_{\bar{r}o}$ of the space $V_{ro} \oplus V_{r\bar{o}} \oplus V_{\bar{r}\bar{o}}$ inside $\mathbb{R}^n$. This complement satisfies

$$\mathbb{R}^n = V_{ro} \oplus V_{r\bar{o}} \oplus V_{\bar{r}o} \oplus V_{\bar{r}\bar{o}} \,, \tag{6}$$

and again $V_{\bar{r}o}$ is not uniquely defined. $V_{\bar{r}o}$ is the space of unreachable and observable states. By picking a basis of $V_{\bar{r}o}$, we obtain a matrix $T_{\bar{r}o}$ whose columns are the basis elements.

Since the direct sum of the spaces V_{ro}, $V_{r\bar{o}}$, $V_{\bar{r}o}$ and $V_{\bar{r}\bar{o}}$ is $\mathbb{R}^n$, the matrix

$$T = [T_{ro}\ T_{r\bar{o}}\ T_{\bar{r}o}\ T_{\bar{r}\bar{o}}]$$

has size $n \times n$ and is invertible.

Taking into account the fact that $\mathcal{B} \subset \mathcal{R}$ and $\mathcal{N} \subset N(C)$, and the A-invariance of $\mathcal{R}$ and $\mathcal{N}$, it is easy to verify that

$$A\,[T_{ro}\ T_{r\bar{o}}\ T_{\bar{r}o}\ T_{\bar{r}\bar{o}}] = [T_{ro}\ T_{r\bar{o}}\ T_{\bar{r}o}\ T_{\bar{r}\bar{o}}] \left[\begin{array}{cc|cc} A_{ro} & 0 & A_{13} & 0 \\ A_{21} & A_{r\bar{o}} & A_{23} & A_{24} \\ \hline 0 & 0 & A_{\bar{r}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{r}\bar{o}} \end{array}\right] \tag{7a}$$

$$B = [T_{ro}\ T_{r\bar{o}}\ T_{\bar{r}o}\ T_{\bar{r}\bar{o}}] \begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \\ \hdashline 0 \\ 0 \end{bmatrix} \tag{7b}$$

$$C\,[T_{ro}\ T_{r\bar{o}}\ T_{\bar{r}o}\ T_{\bar{r}\bar{o}}] = [C_{ro}\quad 0\quad C_{\bar{r}o}\quad 0]\,. \tag{7c}$$

If we consider the transformed state vector

$$\tilde{\underline{x}} = T^{-1}\underline{x} = \begin{bmatrix} \underline{x}_{ro} \\ \underline{x}_{r\bar{o}} \\ \underline{x}_{\bar{r}o} \\ \underline{x}_{\bar{r}\bar{o}} \end{bmatrix}, \tag{8}$$

the system dynamics take the form

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_{ro} \\ \underline{x}_{r\bar{o}} \\ \underline{x}_{\bar{r}o} \\ \underline{x}_{\bar{r}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{ro} & 0 & A_{13} & 0 \\ A_{21} & A_{r\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{r}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{r}\bar{o}} \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} \underline{x}_{ro} \\ \underline{x}_{r\bar{o}} \\ \underline{x}_{\bar{r}o} \\ \underline{x}_{\bar{r}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{\tilde{B}} \underline{u}(t) \tag{9}$$

$$\underline{y}(t) = [\underbrace{C_{ro}\quad 0\quad C_{\bar{r}o}}_{\tilde{C}}\quad 0] \begin{bmatrix} \underline{x}_{ro} \\ \underline{x}_{r\bar{o}} \\ \underline{x}_{\bar{r}o} \\ \underline{x}_{\bar{r}\bar{o}} \end{bmatrix} + D\,\underline{u}(t) \tag{10}$$

where the relations (9)-(10) are a consequence of the similarity transformation rule

$$\tilde{A} = T^{-1}AT\ , \quad \tilde{B} = T^{-1}B \quad \tilde{C} = CT \tag{11}$$

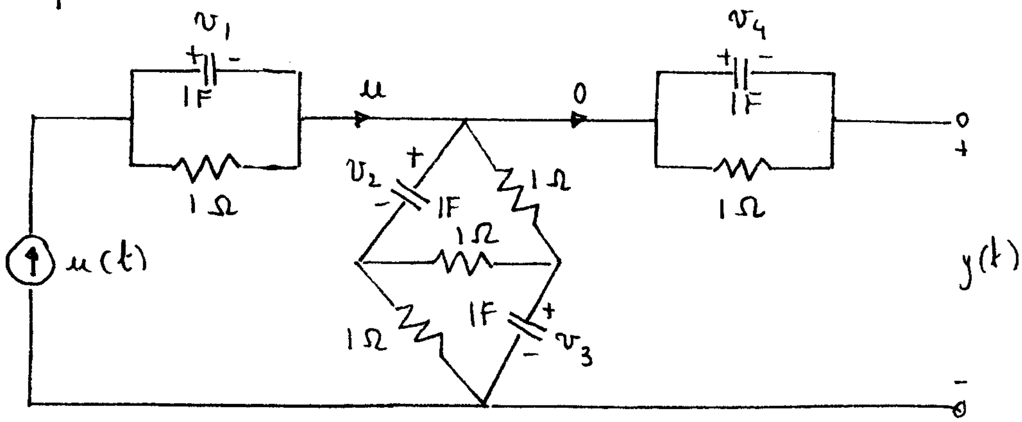
and identities (7a)-(7c).

An important feature of the above decomposition is that only the reachable and observable part needs to be retained to realize the transfer function of the original system, i.e.

$$H(z) = C(zI-A)^{-1}B + D = C_{ro}(zI - A_{ro})^{-1}B_{ro} + D . \qquad (12)$$

In other words, states which are either unreachable or unobservable do not contribute to the transfer function $H(z)$.

<u>Example</u>: Consider the circuit



considered in Lecture 11. Its state-space model is given by

$$\begin{bmatrix} \dot v_1 \\ \dot v_2 \\ \dot v_3 \\ \dot v_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 2/3 \\ 2/3 \\ 0 \end{bmatrix}}_{B} u(t)$$

$$y = \underbrace{\begin{bmatrix} 0 & 2/3 & 2/3 & -1 \end{bmatrix}}_{C} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \underbrace{\frac{1}{3}}_{D} u(t) .$$

Its reachability matrix takes the form

$$R = [B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2/3 & -2/3^2 & 2/3^3 & -2/3^4 \\ 2/3 & -2/3^2 & 2/3^3 & -2/3^4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and its column space $\mathcal{R}$ is spanned by

$$\underline{t}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{t}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It was also shown in Lecture 11 that the right null space $\mathcal{N}$ of the observability matrix is spanned by

$$\underline{t}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \underline{t}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

(i) The intersection $V_{r\bar{o}} = \mathcal{R} \cap \mathcal{N}$ = the reachable and unobservable space is therefore spanned by $\underline{t}_2$.

(ii) A complement V_{ro} of $V_{r\bar{o}}$ inside $\mathcal{R}$ is the space spanned by $\underline{t}_1$, which constitutes the reachable and observable space.

(iii) A complement $V_{\bar{r}\bar{o}}$ of $V_{r\bar{o}}$ inside $\mathcal{N}$ is the space spanned by $\underline{t}_4$, which forms the unreachable and unobservable space.

(iv) Finally, a complement $V_{\bar{r}o}$ of $V_{ro} \oplus V_{r\bar{o}} \oplus V_{\bar{r}\bar{o}}$ inside $\mathbb{R}^4$ is given by the space spanned by

$$\underline{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If we now perform the similarity transformation $\underline{x} = T\,\tilde{\underline{x}}$ with

$$T = [\underline{t}_1\ \underline{t}_2\ \underline{t}_3\ \underline{t}_4] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix} ,$$

the transformed variables take the form

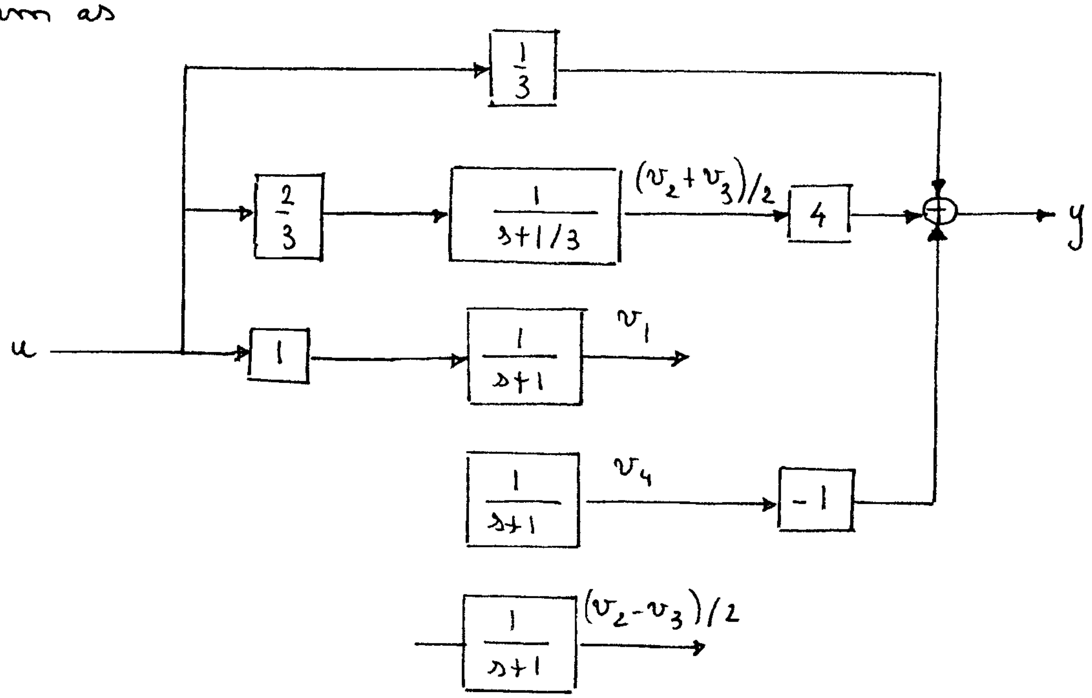
$$T^{-1}\underline{x} = \tilde{\underline{x}} = \begin{bmatrix} (v_2+v_3)/2 \\ v_1 \\ v_4 \\ (v_2-v_3)/2 \end{bmatrix}$$

so that $(v_2+v_3)/2$ is reachable and observable, v_1 is reachable but unobservable, v_4 is observable but unreachable, and $(v_2-v_3)/2$ is unreachable and unobservable, as was already demonstrated in Lecture 11. The transformed dynamics are given by

$$\tilde{C} = CT = [4/3 \ \ 0 \ \ -1 \ \ 0] \qquad \tilde{B} = T^{-1}B = \begin{bmatrix} 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -1/3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \tilde{D} = D = \frac{1}{3}$$

so that the transformed system can be represented in block diagram form as



<u>Definition</u>: A state-space model (A, B, C, D) is said to be a realization of a $p \times m$ transfer function $H(z)$ if

$$H(z) = C(zI - A)^{-1}B + D \ .$$

If this model has the smallest number n of states among all realizations of $H(z)$, (A, B, C, D) is a <u>minimal</u> realization.

The fact that an arbitrary state-space model admits a Kalman decomposition indicates immediately that in order for a realization

(A, B, C, D) to be minimal, it must be both reachable and observable. Otherwise we could construct a realization with fewer states by removin the unreachable or unobservable components of the system.

Another useful observation is that if $H(z)$ admits a finite dimen-sional state-space model (A, B, C, D) (by finite dimensional, we mean that the number n of states is finite), the $p \times m$ transfer matrix $H(z)$ must be <u>rational</u>, i.e. its entries $h_{ij}(z)$, $1 \leq i \leq p$ $1 \leq j \leq m$ can be expressed as

$$h_{ij}(z) = \frac{n_{ij}(z)}{d_{ij}(z)} \tag{13}$$

with $n_{ij}(z)$ and $d_{ij}(z)$ polynomial. This is a consequence of the fact that

$$\begin{aligned} H(z) &= C(zI-A)^{-1}B + D \\ &= \frac{1}{a(z)}\left[C\,\widetilde{(zI-A)}\,B + D\,a(z) \right] \end{aligned} \tag{14}$$

where $a(z) = \det(zI - A)$ and the adjugate matrix $\widetilde{zI-A}$ is polynomial, i.e. all its entries are polynomial. Thus, only rational transfer matrices can admit state-space realizations.

Obviously, the knowledge of the transfer matrix

$$H(z) = \sum_{k=0}^{\infty} H(k)\, z^{-k} \tag{15}$$

is equivalent to the knowledge of the matrix impulse response $\{H(k), k \geq 0\}$ so that the problem of realizing $H(z)$ in state-space form is equivalent to that of realizing the impulse response $\{H(k), k \geq 0\}$. Then, if $(A, B, C,$ is a realization of $\{H(k), k \geq 0\}$, we have

$$H(0) = D \tag{16a}$$

$$H(k) = CA^{k-1}B \quad \text{for } k \geq 1 . \tag{16b}$$

Consider now the block Hankel matrix

$$\mathcal{H}[1,q] = \begin{bmatrix} H(1) & H(2) & H(3) & \cdots & H(q) \\ H(2) & H(3) & & & H(q+1) \\ H(3) & & & & \vdots \\ \vdots & & & & \\ H(q) & H(q+1) & \cdots & & H(2q-1) \end{bmatrix} . \tag{17}$$

The blocks $H(k)$ of this matrix have size $p \times m$, so that $\mathcal{H}[1,q]$ has size $pq \times mq$. The block Hankel property manifests itself by the fact that the blocks $H(k)$ repeat themselves along the antidiagonals of $\mathcal{H}[1,q]$. Then, from (16b) we deduce that

$$\mathcal{H}[1,q] = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{q-1}B \end{bmatrix} = \mathcal{O}_q \mathcal{R}_q \, , \tag{18}$$

i.e. $\mathcal{H}[1,q]$ is the product of the observability and reachability matrices

O_q and R_q. This observation can be employed to prove the following result

<u>Lemma 1</u>: Let (A, B, C, D) be a n-dimensional realization of $\{H(k), k \geq 0\}$. Then, for q arbitrary, the block Hankel matrix $\mathcal{H}[1,q]$ satisfies

$$\text{rank } \mathcal{H}[1,q] \leq n \quad . \tag{19}$$

Furthermore, for $q \geq n$, if (A,B) is reachable and (C,A) is observable, we have

$$\text{rank } \mathcal{H}[1,q] = n \quad . \tag{20}$$

<u>Proof</u>: From the identity (18), where O_q has size $pq \times n$ and R_q has size $n \times mq$, we see that the columns of $\mathcal{H}[1,q]$ can be expressed as linear combinations of the n columns of O_q, where the weights appearing in these combinations are given by the columns of R_q. This means that the column space of $\mathcal{H}[1,q]$ is spanned by the n columns of O_q, so that

$$\text{rank } \mathcal{H}[1,q] \leq n \; .$$

If (C,A) is observable and $q \geq n$, the matrix

$$O_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}$$

has full rank (O_q has the same row space as O_n, which has rank n), so that the n columns of O_q are linearly independent. Furthermore, by multiplying (18) on the right by R_q^T, we obtain

$$\mathcal{H}[1,q]\, R_q^T = O_q\, R_q\, R_q^T \tag{21}$$

where, since (A,B) is reachable, the $n \times n$ reachability Gramian matrix

$$W_q = R_q\, R_q^T \tag{22}$$

is invertible. This implies

$$\mathcal{H}[1,q]\, R_q^T\, (R_q R_q^T)^{-1} = O_q \quad , \tag{23}$$

so that the columns of O_q are also spanned by the columns of $\mathcal{H}[1,q]$. Thus, the column spaces of $\mathcal{H}[1,q]$ and O_q are identical, and since the n columns of O_q are linearly independent, we have

$$\operatorname{rank} \mathcal{H}[1,q] = n \quad . \qquad \blacksquare$$

The above lemma yields the following characterization of minimality.

<u>Theorem</u> 1: A realization (A, B, C, D) of $H(z)$ is minimal if and only if it is reachable and observable.

<u>Proof</u>: As was noted earlier, if a realization is minimal, it must be reachable and observable, since otherwise we could construct a realization of smaller dimension by removing its unreachable or unobservable component. Conversely, let (A, B, C, D) be a n-dimensional reachable and observable

realization of $H(z)$. If $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is another realization of dimension $\bar{n} < n$, we have

$$H(k) = CA^{k-1}B = \bar{C}\bar{A}^{k-1}\bar{B} \quad \text{for } k \geq 1 , \tag{24}$$

so that the Hankel

$$\mathcal{H}[1,n] = O_n R_n = \bar{O}_n \bar{R}_n . \tag{25}$$

Since (A,B) is reachable and (C,A) observable, from Lemma 1 we can conclude that

$$\text{rank } \mathcal{H}[1,n] = n . \tag{26a}$$

Furthermore, since $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is another realization of dimension $\bar{n}$, the Lemma 1 also implies

$$\text{rank } \mathcal{H}[1,n] \leq \bar{n} . \tag{26b}$$

Combining (26a) and (26b) yields $n \leq \bar{n}$, so that if (A,B,C,D) is a reachable and observable realization of $H(z)$, its dimension is minimal among all realizations of $H(z)$. ∎

Given a SISO system with transfer function $H(z)$, the minimality of a realization (A,b,c,d) can also be characterized as follows.

<u>Theorem</u> 2: A realization (A,b,c,d) of $H(z)$ is minimal if and only if the representation

$$H(z) = \frac{b(z)}{a(z)} , \qquad (27)$$

with $a(z) = \det(zI - A)$ and $b(z) = c\,\widetilde{(zI-A)}\,b + d\,a(z)$, is irreducible, i.e. $a(z)$ and $b(z)$ are coprime.

<u>Proof</u>: Suppose that the n-dimensional realization (A, b, c, d) is not minimal. Then, there exists a realization $(\bar{A}, \bar{b}, \bar{c}, d)$ with dimension $\bar{n} < n$, so that if

$$\bar{a}(z) = \det(zI - \bar{A}) \qquad \bar{b}(z) = \bar{c}\,\widetilde{(zI-\bar{A})}\,\bar{b} + d\,\bar{a}(z) ,$$

we have

$$H(z) = \frac{b(z)}{a(z)} = \frac{\bar{b}(z)}{\bar{a}(z)} \qquad (28)$$

where $\bar{n}$ = degree of $\bar{a}(z) < n$ = degree of $a(z)$. This implies that $a(z)$ and $b(z)$ are not coprime. Conversely, if $a(z)$ and $b(z)$ are not coprime, after cancelling common factors, we obtain a reduced representation

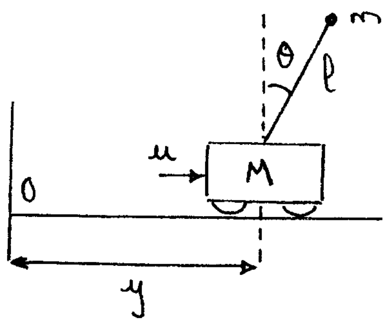
$$H(z) = \frac{\bar{b}(z)}{\bar{a}(z)} \qquad (29)$$

with degree of $\bar{a}(z) = \bar{n} < n$. This reduced representation can be used to realize $H(z)$, say in controller form, with only $\bar{n}$ states, so that the n-dimensional state-space realization (A, b, c, d) of $H(z)$ is not minimal

An interesting consequence of this result is that if a realization

(A, b, c, d) of $H(z)$ with $a(z) = \det(zI - A)$ and $b(z) = c\,\widetilde{(zI-A)}\,b + d\,a(z)$ is such that $H(z) = b(z)/a(z)$ is irreducible, then it is reachable and observable. Conversely, if the polynomials $a(z)$ and $b(z)$ have a common factor, so that there is a pole/zero cancellation, the realization (A, b, c, d) is either not reachable or not observable.

<u>Example</u>: Consider an inverted pendulum mounted on a cart.



M is the mass of the cart, and the mass m of the pendulum is totally concentrated on its end. y is the horizontal position of the cart, and θ the angle of the pendulum rod with respect to the vertical. An external force u is applied to the carriage. In addition, there is a frictional force $-F\dot{y}$ on the cart. The pendulum pivot is frictionless. The acceleration due to gravity is g. Then, for small angles θ, and assuming $\frac{m}{M} \ll 1$, the linearized equations of motion for the carriage and pendulum are given by

$$M\ddot{y} + F\dot{y} = u \qquad (30a)$$

$$\ddot{y} + l\ddot{\theta} = g\theta\,. \qquad (30b)$$

By using the state variables $x_1 = y$, $x_2 = \dot{y}$, $x_3 = y + l\theta$, $x_4 = \dot{y} + l\dot{\theta}$, a state-space model for the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{l} & 0 & \frac{g}{l} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{bmatrix}}_{b} u \qquad (31a)$$

We measure the angle θ of the pendulum with respect to the vertical, so that the output

$$\theta = \underbrace{\begin{bmatrix} -\frac{1}{l} & 0 & \frac{1}{l} & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} . \qquad (31b)$$

By Laplace transforming equations (30a) and (30b) and eliminating the Laplace transform of $z(t)$, we find that the system transfer function is given by

$$\frac{\Theta(s)}{U(s)} = H(s) = \frac{-s}{(Ms+F)(ls^2-g)} \qquad (32)$$

where the degree of the denominator polynomial is only 3. Since the realization (31a)-(31b) has four states, this means it is not minimal. The reachability matrix

$$R = [b \;\; Ab \;\; A^2b \;\; A^3b] = \begin{bmatrix} 0 & \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} \\ \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} & -\frac{F^3}{M^4} \\ 0 & 0 & 0 & -\frac{g}{lM} \\ 0 & 0 & -\frac{g}{lM} & \frac{Fg}{lM^2} \end{bmatrix}$$

has full rank, so that the system is reachable. Then, the presence of a pole/zero cancellation in $H(s)$ implies that the system is not observable (which, of course, can be checked easily). ■

One important property of minimal realizations is that all minimal realizations of a $p \times m$ transfer matrix are equivalent, in the sense that they can be obtained from each other by performing a change of coordinates.

Theorem 3: If (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are two minimal realizations of a $p \times m$ transfer matrix $H(z)$, they are related by a similarity transformation, i.e. there exists an invertible matrix T such that

$$A_2 = T^{-1} A T \quad , \quad B_2 = T^{-1} B \; , \quad , \quad C_2 = C_1 T \quad , \quad D_2 = D_1 \; . \qquad (33)$$

Proof: If (A_i, B_i, C_i, D_i) with $i = 1, 2$ are two minimal realizations of $H(z)$, they have the same size n (otherwise they would not be both minimal). Then, according to (16b), the Hankel matrices $\mathcal{H}[1, n]$ and $\mathcal{H}[2, n+1]$ can be expressed as

$$\mathcal{H}[1, n] = O_1 R_1 = O_2 R_2 \qquad (34)$$

$$\mathcal{H}[2, n+1] = O_1 A_1 R_1 = O_2 A_2 R_2 \; , \qquad (35)$$

where R_i and O_i denote respectively the reachability and observability matrices of the system (A_i, B_i, C_i, D_i) with $i=1,2$. Since both realizations are minimal, the reachability and observability matrices R_i and O_i have full rank, so that in particular for $i=1,2$, the reachability and observability Gramian matrices

$$W_2 = R_2 R_2^T \qquad M_2 = O_2^T O_2 \tag{36}$$

are invertible. Consider now the matrices

$$T = R_1 R_2^T (R_2 R_2^T)^{-1} \tag{37a}$$

$$S = (O_2^T O_2)^{-1} O_2^T O_1 . \tag{37b}$$

By multiplying (34) on the left and on the right by $(O_2^T O_2)^{-1} O_2^T$ and $R_2^T (R_2 R_2^T)^{-1}$, respectively, we find

$$ST = I_n \tag{38}$$

so that the matrix T is invertible and $S = T^{-1}$. Also, multiplying (34) separately by $(O_2^T O_2)^{-1} O_2^T$ on the left, and by $R_2^T (R_2 R_2^T)^{-1}$ on the right we obtain

$$R_2 = T^{-1} R_1 \qquad O_2 = O_1 T , \tag{39}$$

so that considering the first block columns and rows of R_2 and O_2, we find

$$B_2 = T^{-1} B_1 \quad , \quad C_2 = C_1 T . \tag{40}$$

Similarly, by multiplying (35) on the left and on the right by $(O_2^T O_2)^{-1} O_2^T$

and $R_2^T(R_2 R_2^T)^{-1}$, respectively, we obtain

$$A_2 = T^{-1} A_1 T \quad . \tag{41}$$