EEC 250 Linear Systems and Signals

Lecture 13

Topics: a) CT Lyapunov stability

b) Routh-Hurwitz stability test

c) DT Lyapunov stability

A concept of stability which is often used to study state-space models is that of internal, or Lyapunov, stability.

Definition 1: a) The CT system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) \qquad \underline{x}(0) = \underline{x}_0 \tag{1}$$

is stable (S) in the sense of Lyapunov, or stable for short, if for all initial conditions $\underline{x}_0$, there exists a constant $M > 0$ such that

$$\|\underline{x}(t)\| < M \quad \text{for all } t > 0,$$

i.e. the state vector remains bounded for all t.

b) The system (1) is asymptotically stable (AS) if for all initial conditions $\underline{x}(t) \to 0$ as $t \to \infty$.

Let J be the Jordan form of A, i.e.

$$A = T J T^{-1}, \tag{2}$$

where T is the matrix whose columns are the eigenvectors and generalized eigenvectors of A, and

$$J = \begin{bmatrix} J_1 & & & & \\ & \ddots & & 0 & \\ & & J_i & & \\ & 0 & & \ddots & \\ & & & & J_q \end{bmatrix} = \operatorname{diag}\{J_i\} \tag{3a}$$

with

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \tag{3b}$$

where the Jordan block J_i has size $n_i \times n_i$, for $1 \le i \le q$. Then, by performing the similarity transformation $x = Tz$, the system (1) become

$$\dot{\underline{z}}(t) = J\,\underline{z}(t) \qquad \underline{z}(0) = \underline{z}_0 = T^{-1}\underline{x}_0\,, \tag{4}$$

and it is clear that $z(t)$ remains bounded, or tends to zero as $t \to \infty$, if and only if $\underline{x}(t)$ has the same properties. In other words, the stability or asymptotic stability of systems (1) and (4) are equivalent.

Then, by noting that the solution of (4) takes the form

$$\underline{z}(t) = e^{Jt}\,\underline{z}_0 \tag{5}$$

with

$$e^{Jt} = \operatorname{diag}\{e^{J_i t}\} \tag{6a}$$

$$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{n_i - 1}}{(n_i - 1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix} \qquad (6b)$$

we obtain the following stability test.

Theorem 1: a) The system (1) is asymptotically stable if and only if

$$\operatorname{Re} \lambda_i < 0 \quad \text{for all } i, \qquad (7)$$

i.e. all the eigenvalues of A lie in the open left half-plane (the open left half-plane does not include the imaginary axis).

b) The system (1) is stable if and only if

$$\operatorname{Re} \lambda_i \leq 0 \quad \text{for all } i \qquad (8)$$

and the Jordan blocks corresponding to eigenvalues with $\operatorname{Re} \lambda_i = 0$ have size $n_i = 1$. In other words, there are no Jordan blocks of size 2 or more corresponding to eigenvalues on the imaginary axis.

Note that for $\operatorname{Re} \lambda_i = 0$ and $n_i \geq 2$, the quantity

$$|e^{\lambda_i t} t^{n_i - 1}| = t^{n_i - 1} \to \infty \quad \text{as } t \to \infty .$$

One limitation of the above stability test is that it requires the computation of the eigenvalues and Jordan structure of A. Although the eigenvalues of a matrix can be computed reliably, it is sometimes more convenient to employ an indirect approach which does not require the computation of the eigenvalues of A. The Lyapunov theorem described below is the best known of such indirect stability tests.

<u>Theorem 2</u>: Let (A, C) be an observable pair. Then, the matrix A is AS if and only if there exists a symmetric positive-definite matrix P such that

$$A^T P + P A + C^T C = 0 \tag{9}$$

The equation (9) is called Lyapunov's equation.

<u>Proof</u>: <u>Necessity</u>: Since

$$e^{At} = T e^{Jt} T^{-1} \tag{10}$$

where e^{Jt} has the form (6a)-(6b), all the entries of e^{At} are linear combinations of terms of the form $t^k e^{\lambda_i t}$. When A is AS, so that $\mathrm{Re}\,\lambda_i < 0$ for all i, this implies that the matrix

$$P = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{11}$$

is well defined, since all its entries are linear combinations of converging integrals of the form

$$\int_0^\infty t^{k+k'} e^{(\lambda_i+\lambda_j)t}\, dt \, .$$

Note that P corresponds to the observability Gramian matrix of the pair (C, A) over the interval $[0, \infty)$. The observability of (C, A) implies therefore that P is positive definite. To check this, note that if there exists $\underline{x}$ such that $\underline{x}^T P \underline{x} = 0$, we have

$$0 = \underline{x}^T P \underline{x} = \int_0^\infty \|\underline{z}(t)\|^2\, dt \tag{12}$$

with $\underline{z}(t) = C e^{At} \underline{x}$, so that $\underline{z}(t) \equiv 0$ for $0 \le t < \infty$. This implies in particular

$$\begin{bmatrix} \underline{z}(0) \\ \underline{z}^{(1)}(0) \\ \vdots \\ \underline{z}^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \underline{x} = \underline{0} \tag{13}$$

so that $\underline{x} = 0$, hence P is positive definite.

Furthermore

$$\begin{aligned} A^T P + P A &= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} C^T C e^{At} \right) dt \\ &= e^{A^T t} C^T C e^{At} \Big|_{t=0}^{t=\infty} = -C^T C \end{aligned} \tag{14}$$

where we have used the fact that $e^{A^T t} C^T C e^{At} \to 0$ as $t \to \infty$. The matrix P satisfies therefore the Lyapunov equation (12).

Sufficiency: Let P be a symmetric positive definite matrix satisfying (9). Then, let $(\lambda, \underline{x})$ be an arbitrary eigenvalue / eigenvector pair of A, so that

$$A\underline{x} = \lambda \underline{x} .$$

Multiplying (9) on the left by $\underline{x}^H$ and on the right by $\underline{x}$ gives

$$(\lambda + \lambda^*)\ \underline{x}^H P \underline{x} = -\underline{x}^H C^T C \underline{x}$$

Since P is positive definite, we have $\underline{x}^H P \underline{x} > 0$. The observability of (C, A) implies also $C\underline{x} \neq 0$ (the right eigenvectors of A cannot be orthogonal to the rows of C), and thus

$$2 \operatorname{Re} \lambda = \lambda + \lambda^* = -\frac{\| C \underline{x} \|^2}{\underline{x}^H P \underline{x}} < 0 \tag{15}$$

so that A is asymptotically stable. ∎

Another consequence of the asymptotic stability of a matrix is as follows.

Lemma 1: If A is an AS matrix, the Lyapunov equation

$$A^T P + P A + Q = 0 \tag{16}$$

has a unique solution for every matrix Q.

Proof: If P_1 and P_2 are two solutions, let $\Delta P = P_1 - P_2$. We have

$$A^T \Delta P + \Delta P A = 0 ,$$

so that $D(t) = e^{A^T t} \Delta P e^{At}$ satisfies $\frac{dD}{dt} = 0$, so that $D(t)$ is constant.
This implies

$$\Delta P = \lim_{t \to \infty} e^{A^T t} \Delta P e^{At} = 0 \tag{17}$$

where we have used the fact that since A is AS, $e^{At} \to 0$ as $t \to \infty$. ∎

<u>Physical interpretation</u>: Consider the system

$$\dot{\underline{x}} = A \underline{x} \qquad \underline{x}(0) = \underline{x}_0 \tag{18a}$$

$$\underline{y} = C \underline{x} \tag{18b}$$

If P is a positive definite solution of the Lyapunov equation (9), $E(t) = x^T(t) P \underline{x}(t)$ can be viewed as a function measuring the energy stored inside the system (18) at time t. This function is usually called a Lyapunov function. From (9) we find that

$$\frac{dE}{dt} = \underline{x}^T (A^T P + P A) \underline{x} = \underline{x}^T C^T C \underline{x} = - \|\underline{y}\|^2 , \tag{19}$$

so that the energy of the system decreases by the power $\|\underline{y}\|^2$ delivered to the output. In (19), the output $y(t)$ cannot be identically zero for all $t \geq 0$, since this would mean that the system is not observable. Thus $\frac{dE}{dt} < 0$, which implies that $E(t) \to 0$ as $t \to \infty$, and thus $\underline{x}(t) \to 0$ as $t \to \infty$, so that the system is AS.

Since the stored energy of a system is often easy to evaluate, for many physical systems, it is possible to write a Lyapunov function by inspection. For an RLC circuit with I inductors, J capacitors and K resistors, a Lyapunov function is given by

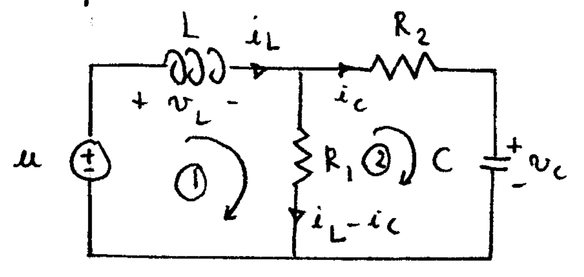
$$E = \frac{1}{2}\left(\sum_{i=1}^{I} L_i i_{L_i}^2 + \sum_{j=1}^{J} C_j v_{C_j}^2\right) \tag{20a}$$

where i_{L_i} and v_{C_j} denote the inductor currents and capacitor voltages, respectively. Furthermore, by observing that the rate of decrease of the energy stored in an RLC circuit corresponds to the power dissipated in its resistors, we see that the identity (19) is satisfied with

$$\|\underline{y}\|^2 = \sum_{k=1}^{K} R_k i_{R_k}^2 \tag{20b}$$

where i_{R_k} denotes the current through the k^{th} resistor.

Example: Consider the circuit



Writing the KVL around loops 1 and 2 gives

$$u = v_L + R_1(i_L - i_c)$$

$$R_1(i_L - i_c) = R_2 i_c + v_c \; .$$

Solving for v_L and i_c yields

$$L\frac{di_L}{dt} = v_L = -\frac{R_1 R_2}{R_1+R_2} i_L - \frac{R_1}{R_1+R_2} v_c + u \tag{21a}$$

$$C\frac{dv_c}{dt} = i_c = \frac{R_1}{R_1+R_2} i_L - \frac{1}{R_1+R_2} v_c \; . \tag{21b}$$

The energy stored in the circuit is $E = \frac{1}{2}(L i_L^2 + C v_c^2)$, and from (21a)-(21b) we find for $u \equiv 0$

$$\frac{dE}{dt} = v_L i_L + v_c i_c = -\frac{R_1 R_2}{R_1 + R_2} i_L^2 - \frac{1}{R_1 + R_2} v_c^2 < 0 \qquad (22)$$

Furthermore, it is easy to check that the power

$$P_R = R_1 (i_L - i_c)^2 + R_2 i_c^2$$

dissipated in resistors R_1 and R_2 equals the right hand side of (22) ■

From the above observations, we can deduce that an RLC circuit is AS if and only if the inductor currents and capacitor voltages are observable from the resistor currents (or voltages). On the other hand, when the inductor currents and capacitor voltages are not observable from the resistor currents, we have

$$\frac{dE}{dt} \leq 0 \qquad (23)$$

so that the stored energy is nonincreasing and the state variables remain bounded, which implies that the circuit is stable (but not AS)

<u>Routh-Hurwitz stability test</u>

The Lyapunov theorem can be used to obtain a proof of the Routh-Hurwitz stability test. The goal of this test is to determine whether the polynomial

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n \qquad (24)$$

has its roots in the left half-plane. The polynomial $a(s)$ can be decomposed as

$$a(s) = p_0(s) + p_1(s) \qquad (25)$$

with

$$p_0(s) = s^n + a_2 s^{n-2} + a_4 s^{n-4} + \dots \qquad (26a)$$

$$p_1(s) = a_1 s^{n-1} + a_3 s^{n-3} + a_5 s^{n-5} + \dots \qquad (26b)$$

Let $a_e(s)$ and $a_0(s)$ be the even and odd parts of $a(s)$, so that

$$a_e(s) = \frac{1}{2}\left(a(s) + a(-s)\right) \qquad a_0(s) = \frac{1}{2}\left(a(s) - a(-s)\right).$$

When n is even, we have

$$p_0(s) = a_e(s) \qquad p_1(s) = a_0(s) .$$

Similarly, when n is odd

$$p_0(s) = a_0(s) \qquad p_1(s) = a_e(s) .$$

Then, the Euclidean division algorithm can be used to generate a sequence of polynomials $p_k(s)$ with $0 \leq k \leq n$, such that

$$p_k(s) = \alpha_{k+1}\, s\, p_{k+1}(s) + p_{k+2}(s) \qquad (27)$$

with degree of $p_k(s) = n-k$ for all k, so that degree of $p_{k+2}(s) <$ degree of $p_{k+1}(s)$. In (27) if p_{k0} and $p_{k+1\,0}$ denote the coefficients of s^{n-k} and

$s^{n-(k+1)}$ in $p_k(s)$ and $p_{k+1}(s)$, respectively, we have

$$\alpha_{k+1} = \frac{p_{k0}}{p_{k+1\,0}} \quad . \tag{28}$$

Furthermore if $p_k(s)$ and $p_{k+1}(s)$ are respectively even and odd, $p_{k+2}(s)$ is even, and if $p_k(s)$ and $p_{k+1}(s)$ are respectively odd and even, $p_{k+2}(s)$ is odd. Note that in (28) we assume $p_{k0} \neq 0$ and thus $\alpha_k \neq 0$ for all k.

The relation (27) can be rewritten as

$$\frac{p_k(s)}{p_{k+1}(s)} = \alpha_{k+1} s + \cfrac{1}{\cfrac{p_{k+1}(s)}{p_{k+2}(s)}} \tag{29}$$

so that the procedure described above to generate the coefficients $\{\alpha_k, 1 \leq k \leq n\}$ is equivalent to performing the continued fraction expansion

$$\frac{p_1(s)}{p_0(s)} = \cfrac{1}{\alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{1}{\alpha_3 s + \cfrac{1}{\ddots\; \cfrac{1}{\alpha_{n-1} s + \cfrac{1}{\alpha_n s}}}}}} \tag{30}$$

In the following, it will be convenient to consider the normalized poly-nomials $D_k(s) = p_k(s)/p_{k0}$, so that the coefficient of s^{n-k} in $D_k(s)$ is one. With this normalization, the recursion (27) takes the form

$$D_k(s) = s\, D_{k+1}(s) + \frac{1}{\alpha_{k+1}\alpha_{k+2}} D_{k+2}(s) \quad . \tag{31}$$

This can be used to prove the following result.

<u>Lemma 2</u>: A state-space realization of the transfer function $G(s) = p_1(s)/p_0(s)$ is given by

$$F = \begin{bmatrix} 0 & -\frac{1}{\alpha_1} & & & & \\ \frac{1}{\alpha_2} & 0 & -\frac{1}{\alpha_2} & & 0 & \\ & \frac{1}{\alpha_3} & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{\alpha_{n-1}} & 0 & -\frac{1}{\alpha_{n-1}} \\ & 0 & & & \frac{1}{\alpha_n} & 0 \end{bmatrix} \qquad g = \begin{bmatrix} \frac{1}{2^{1/2}\alpha_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{32a}$$

$$h = [\, 2^{1/2} \;\; 0 \; \ldots \; 0 \,] \quad . \tag{32b}$$

<u>Proof</u>: Let F_k be the matrix obtained by deleting the first k rows and columns of F, so that

$$sI - F_k = \begin{bmatrix} s & \frac{1}{\alpha_{k+1}} & & & 0 \\ -\frac{1}{\alpha_{k+2}} & s & \frac{1}{\alpha_{k+2}} & & \\ & \ddots & & \ddots & \\ 0 & & -\frac{1}{\alpha_{n-1}} & s & \frac{1}{\alpha_{n-1}} \\ & & & -\frac{1}{\alpha_n} & s \end{bmatrix} . \tag{33}$$

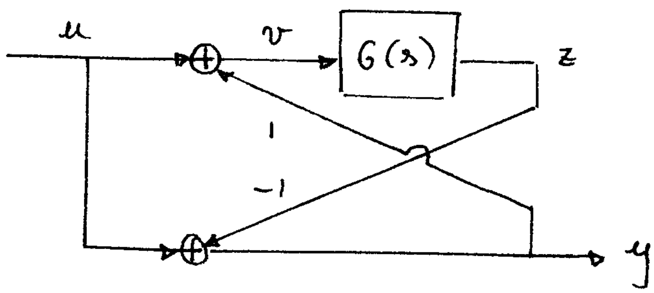
Let $D_k(s) = \det(sI - F_k)$. By expanding $D_k(s)$ with respect to the first row of

$sI-F_k$, we find that it satisfies the recursion (31) with $D_{n-1}(s)=s$ and $D_n(s)=1$. Furthermore, noting that the cofactor of the (1,1) entry of $sI-F$ is $D_1(s)$, we obtain

$$G(s)=h(sI-F)^{-1}g=\frac{D_1(s)}{\alpha_1 D_0(s)}=\frac{P_1(s)}{P_0(s)}. \qquad (34)$$

■

Then, consider the feedback configuration shown below



Observing that

$$y=u-z \qquad z=G(u+y) \qquad (35)$$

we find that the transfer function of the resulting system is

$$\frac{Y(s)}{U(s)}=H(s)=\frac{1-G(s)}{1+G(s)}=\frac{P_0(s)-P_1(s)}{P_0(s)+P_1(s)}=(-1)^n\frac{a(-s)}{a(s)}. \qquad (36)$$

The poles of $H(s)$ are the roots of $a(s)$. Furthermore, since

$$|H(j\omega)|=\left|\frac{a^*(j\omega)}{a(j\omega)}\right|=1, \qquad (37)$$

the system is <u>lossless</u>. From the feedback configuration, starting from the state-space realization

$$\dot{\underline{x}}=F\underline{x}+gv \quad , \quad z=h\underline{x}$$

of $G(s)$, we find that

$$\dot{\underline{x}} = A\underline{x} + bu \tag{38a}$$

$$y = c\underline{x} + du \tag{38b}$$

is a state-space realization of $H(s)$, with

$$A = F - gh = \begin{bmatrix} -\frac{1}{\alpha_1} & -\frac{1}{\alpha_1} & & & & 0 \\ \frac{1}{\alpha_2} & 0 & -\frac{1}{\alpha_2} & & & \\ & \frac{1}{\alpha_3} & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{\alpha_{n-1}} & 0 & -\frac{1}{\alpha_{n-1}} \\ & & & & \frac{1}{\alpha_n} & 0 \end{bmatrix} \qquad b = 2g = \begin{bmatrix} \frac{2^{1/2}}{\alpha_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{39a}$$

$$c = -h = [-2^{1/2}\ 0\ \ldots\ 0] \qquad d = 1 \tag{39b}$$

We are now in position to derive the Routh-Hurwitz stability test.

<u>Theorem 3</u>: The roots of the polynomial $a(s)$ are in the open left half-plan if and only if $\alpha_k > 0$ for all k.

<u>Proof</u>: Since $\det(sI-A) = a(s)$, the roots of $a(s)$ are in the open left half-plane if and only if A is AS. Then if

$$P = \mathrm{diag}\{\alpha_1, \alpha_2 \ldots \alpha_{n-1}, \alpha_n\}, \tag{40}$$

it is easy to verify that P satisfies the Lyapunov equation

$$A^T P + PA + c^T c = 0. \tag{41}$$

The pair (c, A) is observable since the matrix formed by the first n rows of

$$M(s) = \begin{bmatrix} c \\ sI - A \end{bmatrix} = \begin{bmatrix} -2^{1/2} & & & & \\ s + \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & & 0 & \\ -\frac{1}{\alpha_2} & s & \frac{1}{\alpha_2} & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{1}{\alpha_{n-1}} & s & \frac{1}{\alpha_{n-1}} \\ & & & \frac{1}{\alpha_n} & s \end{bmatrix} \tag{42}$$

is lower triangular with nonzero diagonal elements, so that $M(s)$ has full rank for all s. Consequently, by Lyapunov's theorem we see that if the coefficients $\alpha_k > 0$ for all k, P is positive definite, so that A is AS. Conversely, if A is AS, the solution P of the Lyapunov equation (41) is unique and positive definite, so that $\alpha_k > 0$ for all k

<u>Remark</u>: When A is AS, an interesting feature of the realization (A, b, c, d) of $H(s)$ is that it is lossless. Specifically, if $E = \underline{x}^T P \underline{x}$ is the energy function for the system, we have

$$\frac{d}{dt}(\underline{x}^T P \underline{x}) = u^2 - y^2, \tag{43}$$

i.e. the rate of increase in the stored energy equals the power delivered by the input minus that supplied to the output. To verify (43), note that by substitution of (38a) and (38b), this identity is equivalent to

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^T & 0 \\ b^T & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} c^T \\ d \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{44}$$

which can be verified easily.

Routh table: Another way of implementing the Routh-Hurwitz stability test is through the use of the Routh table. The first two rows of this table are the coefficients of $p_0(s)$ and $p_1(s)$. Noting that the leading coefficient of $p_0(s)$ is $a_0 = 1$, we can write $\alpha_1 = \frac{a_0}{a_1}$, so that

$$p_2(s) = p_0(s) - \frac{a_0}{a_1} p_1(s) = p_{20} s^{n-2} + p_{22} s^{n-4} + \ldots \tag{45a}$$

with

$$p_{20} = a_2 - \frac{a_0}{a_1} a_3 \qquad p_{22} = a_4 - \frac{a_0}{a_1} a_5 \quad \ldots, \tag{45b}$$

and the coefficients p_{2k} of $p_2(s)$ constitute the third row of the Routh table. Subsequent rows are constructed in the same way, thus yielding a table of the form shown below

$p_0(s)$	$a_0 = 1$	a_2	a_4	$\ldots$	
$p_1(s)$	a_1	a_3	a_5	$\ldots$	$\alpha_1 = \frac{a_0}{a_1}$
$p_2(s)$	p_{20}	p_{22}	p_{24}	$\ldots$	$\alpha_2 = \frac{a_1}{p_{20}}$
$p_3(s)$	p_{30}	p_{32}	p_{34}	$\ldots$	$\alpha_3 = \frac{p_{20}}{p_{30}}$
$\vdots$	$\vdots$				$\vdots$
$p_n(s)$	p_{n0}				$\alpha_n = \frac{p_{n-1\,0}}{p_{n0}}$

Routh table

The coefficients of the first column of the Routh table are $a_0 = 1$, a_1, $p_{20} \cdots p_{k0} \cdots p_{n0}$. Observing that $\alpha_1 = \frac{a_0}{a_1} \cdots \alpha_k = \frac{p_{k-1\,0}}{p_{k0}} \cdots$ we find that the coefficients α_k are positive (and thus $a(s)$ has its roots in the open LHP) if and only if the coefficients appearing in the first column of the Routh table are all positive.

<u>Example</u>: Consider the polynomial

$$a(s) = (s+3)(s^2 - 2s + 10) = s^3 + s^2 + 4s + 30$$

whose roots are $s_0 = -3 \quad s_\pm = 1 \pm 3j$, so that it is unstable. We have

$$p_0(s) = s^3 + 4s \qquad p_1(s) = s^2 + 30$$

so that

$$p_2(s) = (s^3 + 4s) - s(s^2 + 30) = -26s$$

$$p_3(s) = (s^2 + 30) - \left(-\frac{1}{26}\right) s(-26s) = 30 .$$

The Routh table is therefore given by

$p_0(s)$	1	4
$p_1(s)$	1	30
$p_2(s)$	-26	
$p_3(s)$	30	

Since the first column includes the negative coefficient -26, $a(s)$ is unstable. Incidentally, the number of sign changes in the first column of the Routh table equals the number of RHP roots of $a(s)$. Here there are two such sign changes corresponding to the two RHP roots $s_\pm$.

<u>DT Lyapunov stability</u>: The concept of stability can be extended in a straightforward manner to DT systems of the form

$$\underline{x}(k+1) = A\,\underline{x}(k) \quad , \quad \underline{x}(0) = \underline{x}_0 . \qquad (46)$$

The system (46) is said to be stable (S) in the sense of Lyapunov if for al initial conditions $\underline{x}_0$, there exists a constant $M > 0$ such that

$$\|\underline{x}(k)\| < M \quad \text{for all } k .$$

Similarly, (46) is asymptotically stable (AS) if for all initial condition $\underline{x}(k) \to 0$ as $k \to \infty$.

Observing that $\underline{x}(k) = A^k \underline{x}_0$, with $A^k = T J^k T^{-1}$, where J is the Jordan form of A, it is easy to verify that the system (46) is AS if and only if the eigenvalues λ_i of A are such that $|\lambda_i| < 1$ for all i. Furthermore (46) is S if and only if $|\lambda_i| \leq 1$ for all i, and there ar no Jordan blocks of size 2 or more corresponding to eigenvalues on the unit circle.

Thus, except for the fact that the domain of stability is now the unit disk, the DT and CT stability theories are identical. AS can also be characterized through the DT counterpart of Lyapunov's theorem.

<u>Theorem 4</u>: Let (C, A) be observable. Then the DT system (46) is AS if and only if there exists a symmetric positive-definite matrix P such that

$$P = A^T P A + C^T C \quad . \tag{47}$$

<u>Proof: Necessity</u>: If A is AS, the matrix

$$P = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k \tag{48}$$

is well defined and corresponds to the observability Gramian of the pair (C, A) over the interval $[0, \infty)$. The observability of (C, A) implies that P is positive definite, since otherwise there would be a nonzero vector $\underline{x}$ such that

$$0 = \underline{x}^T P \underline{x} = \sum_{k=0}^{\infty} \| C A^k \underline{x} \|^2 \tag{49}$$

which implies that $CA^k \underline{x} \equiv 0$ for all k, so that $\underline{x}$ would be unobservable, a contradiction. Finally, if P is given by (48), it is easy to check it satisfies (47).

<u>Sufficiency</u>: If P is a positive definite solution of (47), let $(\lambda, \underline{x})$ be an arbitrary eigenvalue/eigenvector pair of A. By multiplying (47) on the left and right by $\underline{x}^H$ and $\underline{x}$, respectively, we have

$$(1 - |\lambda|^2)\, \underline{x}^H P \underline{x} = \| C \underline{x} \|^2 \quad . \tag{50}$$

In this identity $\underline{x}^H P \underline{x} > 0$ since P is positive definite. Also, the observability of (C, A) implies that $C\underline{x} \neq 0$. This implies $1 - |\lambda|^2 > 0$, so that

λ is inside the unit circle. ∎