

EECS 250 Linear Systems and Signals

Lecture 15

- Topics:
- Asymptotic observers
 - Joint controller/observer
 - Reduced-order observers

Motivation: In the last lecture, when designing a full-state feedback controller

$$\underline{u} = \underline{v} - K\underline{x} \quad (1)$$

for a dynamical system of the form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad \underline{x}(0) = \underline{x}_0, \quad (2)$$

it was assumed that all states could be measured directly. Unfortunately this assumption is usually unrealistic. Certain systems have state-space models which include more than one hundred states, and purchasing sensors to measure all of them would be prohibitively expensive. Other systems contain components which operate in an extremely hot or highly toxic environment where it is difficult to introduce sensors. The states associated to such components need in general to be estimated indirectly by measuring more accessible variables.

This suggests that a more sensible way to design feedback controllers consists in employing only a limited set of measurements

$$\underline{y} = C\underline{x} + D\underline{u} \quad (3)$$

to estimate the state vector \underline{x} . Given the resulting estimate $\hat{\underline{x}}$, it is then substituted for the true state \underline{x} in the feedback law (1), thus yielding the controller

$$\underline{u} = \underline{v} - K\hat{\underline{x}} \quad (4)$$

This procedure will be effective as long as: (i) the estimate $\hat{\underline{x}}$ tracks accurately the state \underline{x} , (ii) the substitution of \underline{x} by $\hat{\underline{x}}$ in the feedback law (4) does not affect significantly the closed-loop dynamics of the system. Somewhat surprisingly it turns out that both of these objectives can be achieved.

Asymptotic observers: To estimate \underline{x} , we assume that the model (A, B, C, D) is given, but the initial state \underline{x}_0 is unknown. Then, the simplest procedure one can conjure to estimate \underline{x} consists in running a copy

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} \quad (5)$$

$$\hat{\underline{y}} = C\hat{\underline{x}} + D\underline{u} \quad (6)$$

of the given system (2)-(3), where the initial condition $\hat{\underline{x}}(0)$ represents

our best "guess" of the initial state x_0 . In this case, if $\tilde{x} = x - \hat{x}$ represents the state estimation error, by subtracting (5) from (2) we find

$$\dot{\tilde{x}} = A \tilde{x} \quad , \quad \tilde{x}(0) = x_0 - \hat{x}(0) \quad (7)$$

The relation (7) shows that the error dynamics are totally governed by A . In particular, if A is unstable, the tracking error \tilde{x} will diverge. Even if A is stable, we have no control over the rate of decay of \tilde{x} . This unsatisfactory feature of the estimator (5)-(6) is due to the fact that it is an open-loop observer, i.e. it does not adjust its dynamics to take into account the output tracking error

$$e = y - \hat{y} = C \tilde{x} \quad (8)$$

Since the output tracking error $e = y - \hat{y}$ can be measured, we can feed it back into the observer dynamics (5), thus yielding a closed-loop observer of the form

$$\dot{\hat{x}} = A \hat{x} + B u + L e \quad , \quad (9)$$

where L denotes the gain matrix for the output error e . In this case, the error dynamics become

$$\dot{\tilde{x}} = (A - LC) \tilde{x} \quad , \quad \tilde{x}(0) = x_0 - \hat{x}(0) \quad (10)$$

and the corresponding characteristic polynomial is given by

$$\beta(s) = \det (sI - A + LC) \quad (11)$$

This leads to the question: how does the gain matrix L affect the error characteristic polynomial $\beta(s)$? Since the determinant of a matrix is not affected by matrix transposition, we have

$$\beta(s) = \det (sI - A^T + C^T L^T) \quad (12)$$

which except for the substitution

$$A \rightarrow A^T \quad B \rightarrow C^T \quad K \rightarrow L^T \quad (13)$$

is similar to the problem of selecting a feedback gain matrix K to adjust the characteristic polynomial

$$\alpha(s) = \det (sI - A + BK) \quad (14)$$

of a full-state feedback controller. This suggests that the properties of closed-loop asymptotic observers are similar to those of full state feedback controllers. To study these properties, it is convenient to rewrite the closed-loop observer dynamics (3), (6) as

$$\dot{\hat{x}} = (A - LC) \hat{x} + (B - LD) u + Ly \quad (15a)$$

$$\hat{y} = C \hat{x} + D u \quad (15b)$$

The Laplace transform of the observer output \hat{y} can be expressed in terms of the input Laplace transforms $\underline{U}(s)$ and $\underline{Y}(s)$ as

$$\hat{\underline{Y}}(s) = G_u(s) \underline{U}(s) + G_y(s) \underline{Y}(s) \quad (16a)$$

with

$$G_U(s) = C(sI - A + LC)^{-1} (B - LD) + D \quad (16b)$$

$$G_Y(s) = C(sI - A + LC)^{-1} L. \quad (16c)$$

Observer properties: 1) If (C, A) is observable, so is $(C, A - LC)$. In other words, observability is not affected by output injection in the state dynamics. This is due to the fact that

$$\begin{bmatrix} sI - A + LC \\ C \end{bmatrix} = \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A \\ C \end{bmatrix}, \quad (17)$$

so that if $[sI - A^T \ C^T]^T$ has full column rank, so does $[sI - (A - LC)^T \ C^T]^T$.

2) The zeros of $G_U(s)$ are the same as those of the transfer function $H(s) = C(sI - A)^{-1}B + D$ of the given system. We assume that the number m of system inputs equals the number p of outputs. We have

$$\begin{bmatrix} I & L \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} sI - A + LC & -(B - LD) \\ C & D \end{bmatrix} \quad (18)$$

so that

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \det \begin{bmatrix} sI - A + LC & -(B - LD) \\ C & D \end{bmatrix}, \quad (19)$$

which proves that the zeros of a system are unaffected by output injection.

3) The unobservable modes of the pair (C, A) are unaffected by output injection. To prove this, it is convenient to assume that (C, A) has been decomposed into its observable and unobservable components, so that

$$A = \left[\begin{array}{c|c} A_0 & 0 \\ \hline A_{\bar{0}0} & A_{\bar{0}} \end{array} \right] \quad C = [C_0 \ 0] \quad (20)$$

Then, if we partition $L = \begin{bmatrix} L_0 \\ L_{\bar{0}} \end{bmatrix}$ accordingly, we have

$$A - LC = \left[\begin{array}{c|c} A_0 - L_0 C_0 & 0 \\ \hline A_{\bar{0}0} - L_{\bar{0}} C_0 & A_{\bar{0}} \end{array} \right] \quad (21)$$

so that

$$\begin{aligned} \beta(s) &= \det(sI - A + LC) \\ &= \det(sI - A_0 + L_0 C_0) \det(sI - A_{\bar{0}}). \end{aligned} \quad (22)$$

Thus, if (C, A) is not observable, the unobservable modes of the system are not affected by the tracking error gain L , so that we cannot use L to assign arbitrarily the characteristic polynomial $\beta(s)$ of the observer error dynamics.

Observer dynamics assignment: Based on the previous discussion, we see that in order to be able to assign arbitrarily the observer dynamics, the pair (C, A) must be observable. The correspondence (13) indicates also that under the transposition rule of signal flow graphs discussed in Lecture 6, output injection and state feedback are transformed into each other. Specifically, the transposition rule exchanges output nodes into input nodes, and vice-versa, and reverses the direction of each arc in the graph. Thus, injection of an output node into a state node becomes an injection of a state node into an input node, i.e. state feedback. If (C, A) is observable, the pair (A^T, C^T) is reachable. Consequently, selecting a matrix L such that

$$\begin{aligned} \det(sI - A + LC) &= \beta(s) \\ &= s^n + \beta_1 s^{n-1} + \dots + \beta_n, \end{aligned} \tag{23}$$

where $\beta(s)$ is an arbitrary polynomial of degree n , is equivalent to finding a state-feedback gain matrix L^T such that (12) is satisfied.

All the techniques developed in Lecture 14 for the pole assignment problem are therefore directly applicable to the observer dynamics assignment problem. For the case of a single output system, the observer gain is an $n \times 1$ vector

$$l^T = [l_1, l_2, \dots, l_n]$$

Then if

$$\det(sI - A) = a(s) = s^n + a_1 s^{n-1} + \dots + a_n,$$

and

$$\underline{a}^T = [a_1, a_2, \dots, a_n]$$

$$\underline{\beta}^T = [\beta_1, \beta_2, \dots, \beta_n]$$

denote the vectors of coefficients of the open-loop and closed-loop observer error characteristic polynomials, by applying the single input feedback gain matrix formula, we have

$$l^T = (\underline{\beta}^T - \underline{a}^T) R_c R^{-1} \quad (24)$$

where

$$R_c = \begin{bmatrix} 1 & a_1 & & a_{n-1} \\ & \ddots & \ddots & \vdots \\ 0 & & & a_1 \\ & & & 1 \end{bmatrix}^{-1} \quad (25)$$

is the reachability matrix of the controller pair (A_c, b_c) and $R = [C^T A^T C^T \dots (A^T)^{n-1} C^T]$ is the reachability matrix of the pair (A^T, C^T) . Transposing (24) gives

$$\underline{l} = O^{-1} O_0 (\underline{\beta} - \underline{a}) \quad (26)$$

where $O_0 = R_c^T$ is the observability matrix of the observer pair

$(A_0 = A_c^T, c_0 = b_c^T)$ and

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (27)$$

is the observability matrix of (C, A) .

Of course, for low order single output systems, we can also employ the direct approach consisting of expressing the coefficients of the characteristic polynomial $\beta(s) = \det(sI - A + lC)$ in terms of the entries of l .

Example: Consider the inverted pendulum

$$\frac{d}{dt} \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}}_A \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_B u$$

$$y = \underbrace{[1 \ 0]}_C \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}$$

with open-loop characteristic polynomial

$$\det(sI - A) = s^2 - g = (s+3)(s-3).$$

Suppose that we want to design an asymptotic observer with two poles at -2 , so that $\beta(s) = (s+2)^2 = s^2 + 4s + 4$. Then

$$A - p_c = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -p_1 & 1 \\ 9 - p_2 & 0 \end{bmatrix}$$

so that

$$sI - A + p_c = \begin{bmatrix} s + p_1 & -1 \\ p_2 - 9 & s \end{bmatrix}$$

and

$$B(s) = s^2 + p_1 s + (p_2 - 9).$$

Matching coefficients yields

$$p_1 = 4 \quad p_2 = 13.$$

Alternately, if we apply the formula (26), we have

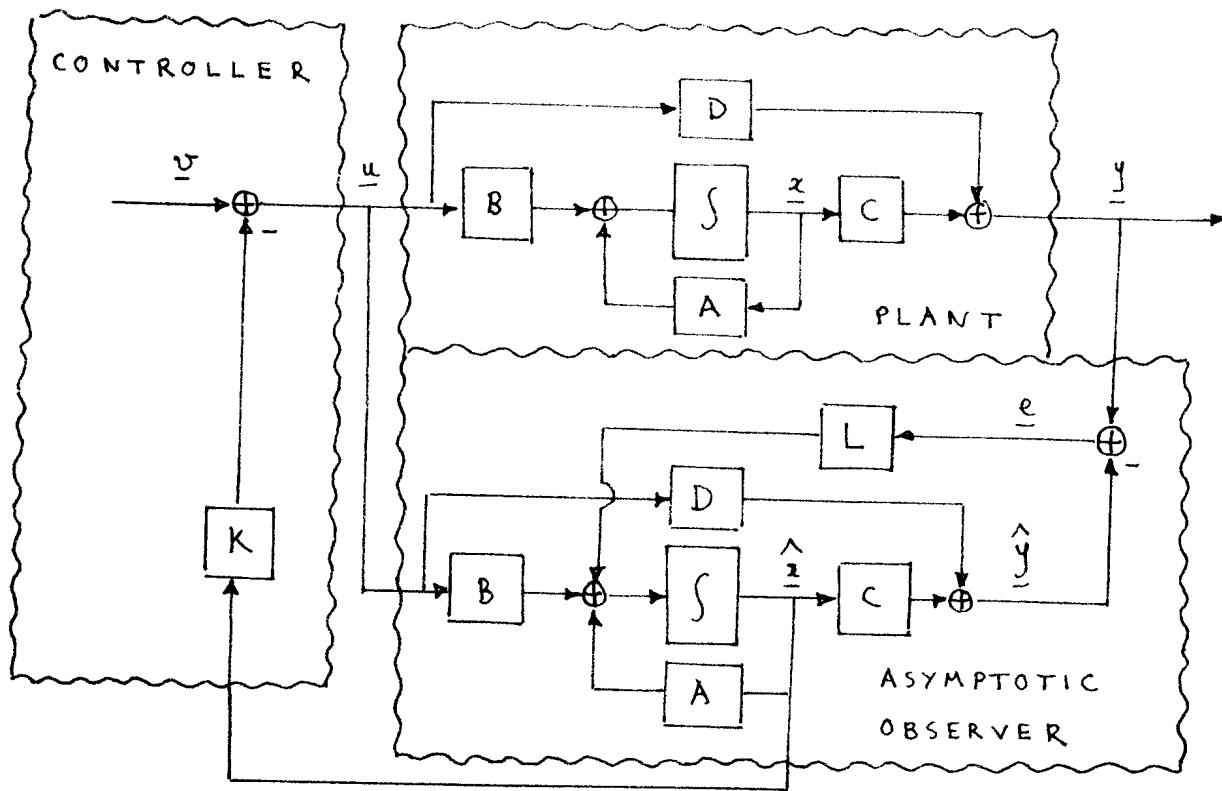
$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = I_2 \quad O_0 = I_2$$

so that

$$\underline{p} = \underline{\beta} - \underline{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}.$$

Thus, both approaches yield the same result, as expected. ■

Joint controller/observer: When the controller $\underline{u} = \underline{v} - K \hat{\underline{x}}$ is combined with an asymptotic observer, the block diagram of the resulting joint controller/observer looks as follows.



The dynamics for the joint controller and observer are given by

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \quad (28a)$$

$$y = \begin{bmatrix} C & -DK \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + D \underline{u} \quad (28b)$$

with initial conditions $[\underline{x}_0^T \hat{\underline{x}}^T(0)]^T$. However, to study the properties of the joint controller/observer, instead of writing the dynamics in terms of \underline{x} and $\hat{\underline{x}}$, it is convenient to express them in terms of \underline{x} and $\tilde{\underline{x}} = \underline{x} - \hat{\underline{x}}$.

This gives

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\tilde{\underline{x}}} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} \underline{x} \\ \tilde{\underline{x}} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u} \quad (29a)$$

$$y = \begin{bmatrix} C-DK & DK \end{bmatrix} \begin{bmatrix} \underline{x} \\ \tilde{\underline{x}} \end{bmatrix} + D \underline{u} \quad (29b)$$

In the following, we assume that (A, B) is reachable, and (C, A) observable, since these assumptions are necessary to assign arbitrarily the controller and observer dynamics, respectively. Then $(A - BK, B)$ is reachable, and the dynamics (29a) are exactly in the form of the decomposition of a system into reachable and unreachable parts, where \underline{x} and $\tilde{\underline{x}}$ are respectively the reachable and unreachable components of the joint controller and observer.

Since the unreachable part of a system drops out of its transfer function, the transfer function of the joint controller/observer system can be expressed as

$$H(s) = (C - DK)(sI - A + BK)^{-1}B + D, \quad (30)$$

which is exactly the transfer function obtained by applying the full state feedback law $\underline{u} = \underline{v} - K\underline{x}$ to the dynamical system (2). Thus, the joint controller/observer system behaves as if the observer was not present, and as if the full state vector \underline{x} was available to implement the controller. Intuitively, this is due to the fact that after selecting L such that $A - LC$ is stable, the error $\tilde{\underline{x}}(t) \rightarrow 0$ as $t \rightarrow \infty$, and thus asymptotically, there is actually no difference between the true state $\underline{x}(t)$ and its estimate $\hat{\underline{x}}(t)$.

According to (29a), we see also that the characteristic polynomial of the joint controller/observer is given by

$$a_{o-c}(s) = \det \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + LC \end{bmatrix}$$
$$= \det(sI - A + BK) \det(sI - A + LC) = \alpha(s) \beta(s), \quad (31)$$

i.e. it is the product of the controller and observer characteristic polynomials. Thus, the controller and observer do not interfere with each other and can be designed separately. This property is called the separation principle of controller and observer design. It carries over to the more general context of stochastic optimal control, where given a set of noisy measurements, the optimum controller minimizing the expected value of a certain objective function can be implemented by first designing a deterministic controller minimizing the objective function based on the knowledge of the full state, and then combining it with an optimum stochastic observer which estimates the state from the given measurements.

Combining (31) with our earlier observation that \tilde{x} is unreachable, we see that the observer poles, i.e. the zeros of $\beta(s)$, are the hidden

modes of the joint controller/observer. This is yet one more manifestation of the fact that the joint controller/observer behaves as if the observer was not present. Note that although the expressions (30) and (31) for the transfer functions and characteristic polynomial of the joint controller and observer were derived by writing this system in its $\underline{x}, \underline{\tilde{x}}$ coordinates, they remain valid in any coordinate system, since the transfer function and characteristic polynomial of a state-space model are invariant under similarity transformations. This is in particular the case of the $\underline{x}, \hat{\underline{x}}$ coordinate system, which is related to the $\underline{x}, \underline{\tilde{x}}$ coordinates through the transformation

$$\begin{bmatrix} \underline{x} \\ \underline{\tilde{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \quad (32)$$

Reduced-order observers: One slightly disconcerting aspect of the asymptotic observers that we have designed up to this point is that they seek to estimate all states from the knowledge of the m inputs \underline{u} and p outputs \underline{y} , even though the relation

$$\underline{y} - \mathbf{D}\underline{u} = \mathbf{C}\underline{x} \quad (33)$$

indicates that p linear combinations of the states are known exactly and need not be estimated. This leads to the concept of reduced-order observer

where instead of estimating all states \underline{x} , we estimate only the $n-p$ states which are not directly available from (33). The design of a reduced-order observer can be performed in two steps.

Step 1: Select a similarity transformation T such that

$$\tilde{C} = CT = [I_p \ 0] . \quad (34)$$

This can be achieved by selecting $n-p$ row vectors \underline{s}_i^T , $p+1 \leq i \leq n$, which together with the rows of C form a basis of \mathbb{R}^n . Then the matrix

$$S = \begin{bmatrix} C \\ \underline{s}_{p+1}^T \\ \vdots \\ \underline{s}_n^T \end{bmatrix} \quad (35)$$

is invertible, and $T = S^{-1}$ satisfies (34). Under this similarity transformation, the new state vector can be partitioned as

$$T^{-1}\underline{x} = \underline{x}' = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \quad (36a)$$

where \underline{x}_1 and \underline{x}_2 have dimensions p and $n-p$, respectively. Partitioning the transformed dynamics accordingly gives

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \tilde{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} . \quad (36b)$$

Step 2: For simplicity, we assume that the input feedthrough matrix $D=0$. Then, substituting the observation $y = \underline{x}_1$, inside the transformed system

$$\dot{\underline{x}}_1 = A_{11} \underline{x}_1 + A_{12} \underline{x}_2 + B_1 \underline{u} \quad (37a)$$

$$\dot{\underline{x}}_2 = A_{21} \underline{x}_1 + A_{22} \underline{x}_2 + B_2 \underline{u} \quad (37b)$$

gives the reduced system

$$\dot{\underline{x}}_2 = \underbrace{A_{22}}_{A_r} \underline{x}_2 + \underbrace{(A_{21} \underline{y} + B_2 \underline{u})}_{\underline{u}_r} \quad (38a)$$

$$\underline{y}_r \triangleq \underline{y} - A_{11} \underline{y} - B_1 \underline{u} = \underbrace{A_{12}}_{C_r} \underline{x}_2 \quad (38b)$$

whose state \underline{x}_2 has only dimension $n-p$, and where $\underline{u}_r, \underline{y}_r$ denote respectively the reduced system input and output.

For this system, we can design a standard asymptotic observer of the form

$$\dot{\hat{\underline{x}}}_2 = A_r \hat{\underline{x}}_2 + \underline{u}_r + L_r (\underline{y}_r - C_r \hat{\underline{x}}_2) \quad (39)$$

where the gain matrix L_r is selected such that

$$\begin{aligned} \det(sI_{n-p} - A_r + L_r C_r) &= \gamma(s) \\ &= s^{n-p} + \gamma_1 s^{n-(p+1)} + \dots + \gamma_{n-p}, \end{aligned} \quad (40)$$

i.e. $\gamma(s)$ is a specified polynomial of degree $n-p$. Subtracting (39) from (38a), we see that the reduced estimator error $\tilde{\underline{x}}_2 = \underline{x}_2 - \hat{\underline{x}}_2$ satisfies

$$\dot{\tilde{x}}_2 = (A_{22} - L_n C_n) \tilde{x}_2 \quad (41)$$

so that $\gamma(s)$ is the characteristic polynomial of the reduced error dynamics.

One difficulty associated with the implementation (39) of the reduced-order observer is that the reduced observation \underline{y}_r contains \underline{y} , so that we need apparently to differentiate the measurement vector \underline{y} . Such a differentiation is highly undesirable, since it tends to introduce a large amount of high frequency noise. To overcome this problem, we can introduce the transformed observer vector

$$\underline{q} = \hat{\underline{x}}_2 - L_n \underline{y} \quad (41)$$

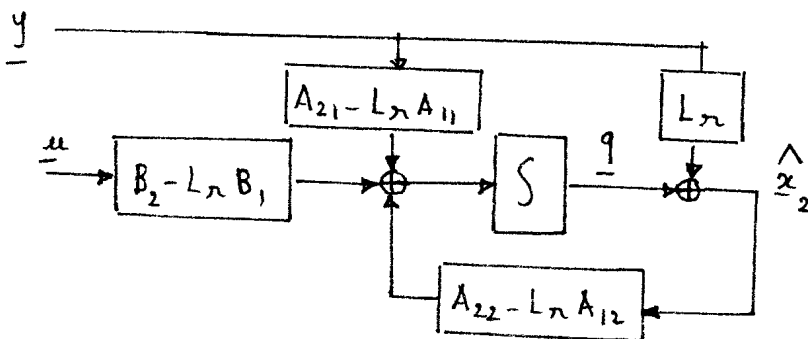
and implement (39) as

$$\begin{aligned} \dot{\underline{q}} &= A_{22} \hat{\underline{x}}_2 + \underline{u}_r - L_n (A_{11} \underline{y} + B_1 \underline{u} + C_n \hat{\underline{x}}_2) \\ &= (A_{22} - L_n A_{12}) \hat{\underline{x}}_2 + (A_{21} - L_n A_{11}) \underline{y} + (B_2 - L_n B_1) \underline{u} \end{aligned} \quad (42)$$

with

$$\hat{\underline{x}}_2 = \underline{q} + L_n \underline{y}, \quad (43)$$

which corresponds to the block diagram shown below.



Finally, in order to be able to select a reduced order observer gain matrix L_n such that the characteristic polynomial $\gamma(s)$ can be selected arbitrarily in (40), the pair $(C_r = A_{12}, A_r = A_{22})$ must be observable. It turns out that this is a consequence of the observability of (C, A) , as shown by the following lemma.

Lemma: (A_{12}, A_{22}) is observable if and only if (C, A) is observable.

Proof: Since similarity transformations do not affect observability, (C, A) is observable if and only if the pair (\tilde{C}, \tilde{A}) given by (34), (36b) is observable, or equivalently, if the matrix

$$\begin{bmatrix} sI - \tilde{A} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \\ I_p & 0 \end{bmatrix} \quad (44)$$

has full rank for all s . But elementary row operations do not affect the rank of a matrix. Using the I_p block in the first column to eliminate the blocks located above it yields

$$\begin{bmatrix} 0 & -A_{12} \\ 0 & sI - A_{22} \\ I_p & 0 \end{bmatrix} \quad (45)$$

which has full rank for all s whenever the submatrix $\begin{bmatrix} -A_{12} \\ sI - A_{22} \end{bmatrix}$ has full

rank for all s , i.e. iff (A_{12}, A_{22}) is observable.

Once \hat{x}_2 is obtained, taking into account $x_1 = y$, the estimate \hat{x} to be employed in the feedback law (4) is given by

$$\hat{x} = T \begin{bmatrix} y \\ \hat{x}_2 \end{bmatrix} \quad (46)$$

Example: Consider the system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with observations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and suppose we want to design a reduced-order observer with a single pole at $s = -3$. Selecting

$$S = T^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so that} \quad T = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

we find

$$\tilde{A} = T^{-1}AT = \begin{array}{c} \overbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}^{A_{11}} \quad \overbrace{\begin{bmatrix} -8 \\ -3 \end{bmatrix}}^{A_{12} = C_2} \\ \hline \overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}^{A_{21}} \quad \overbrace{\begin{bmatrix} -2 \end{bmatrix}}^{A_{22} = A_2} \end{array}$$

$$\tilde{B} = T^{-1}B = \left. \begin{array}{c} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 1 \end{bmatrix} \end{array} \right\} \begin{array}{l} B_1 \\ B_2 \end{array}$$

$$\tilde{C} = CT = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right]$$

Then,

$$A_{22} - L_2 C_2 = -2 - [l_{n_1} \ l_{n_2}] \begin{bmatrix} -8 \\ -3 \end{bmatrix}$$

so that to obtain a reduced-order observer pole at -3, we must have

$$8l_{n_1} + 3l_{n_2} = -1$$

Selecting $l_{n_1} = 0$, $l_{n_2} = -1/3$ gives

$$\dot{\hat{q}} = -3 \hat{x}_2 + [0 \ 4/3] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + [1/3 \ 4/3] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\hat{x}_2 = \hat{q} + [0 \ -1/3] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$