EEC 250 Linear Systems and Signals

Lecture 5

Topics: a) Linear state-space models

b) State-space equations for a linear circuit

c) Realization of a scalar linear differential equation

Linear state-space models: Depending on whether we consider continuous time (CT) or discrete-time (DT) linear systems, the models that we shall study in this course have the form

$$\text{CT:} \quad \begin{aligned} \dot{\underline{x}}(t) &= A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \\ \underline{y}(t) &= C(t)\,\underline{x}(t) + D(t)\,\underline{u}(t) \end{aligned} \qquad (1)$$

with t real, or

$$\text{DT:} \quad \begin{aligned} \underline{x}(k+1) &= A(k)\,\underline{x}(k) + B(k)\,\underline{u}(k) \\ \underline{y}(k) &= C(k)\,\underline{x}(k) + D(k)\,\underline{u}(k) \end{aligned} \qquad (1')$$

with k integer. Here

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n , \quad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m \quad \text{and} \quad \underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p$$

denote respectively the state, input and output vectors. It is easy to check that relations (1) or (1') define a linear system, since a system

S is linear whenever for two arbitrary input/output pairs $\underline{u}_i(t) \xrightarrow{S} \underline{y}_i(t)$ $i=1,2$ and two arbitrary real numbers a, b, then

$$a\underline{u}_1(t) + b\underline{u}_2(t) \xrightarrow{S} a\underline{y}_1(t) + b\underline{y}_2(t) \qquad (2)$$

is also an input/output pair.

In the models (1) and (1') the matrices A, B, C and D have sizes $n \times n$, $n \times m$, $p \times n$ and $p \times m$, respectively. When these matrices are <u>constant</u>, i.e. when they do not depend on the time index t or k, it is easy to verify that the system S is <u>time-invariant</u>. To see this, recall that S is time-invariant if for an arbitrary input/output pair $\underline{u}(t) \xrightarrow{S} \underline{y}(t)$ and an arbitrary time-shift T, then

$$\underline{u}(t-T) \xrightarrow{S} \underline{y}(t-T) \qquad (3)$$

is also an input/output pair. This is due to the fact that if A, B, C, D are constant, the relations (1) remain unchanged if we replace t by t-T, i.e.

$$\dot{\underline{x}}(t-T) = A\,\underline{x}(t-T) + B\,\underline{u}(t-T)$$

$$\underline{y}(t-T) = C\,\underline{x}(t-T) + D\,\underline{u}(t-T) .$$

Although time-varying systems arise in some circumstances, in the following we shall focus our attention on the time-invariant case, i.e.

$$\text{CT}: \quad \begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x}(t) + B\,\underline{u}(t) \\ \underline{y}(t) &= C\,\underline{x}(t) + D\,\underline{u}(t) \end{aligned} \tag{4}$$

or

$$\begin{aligned} \underline{x}(k+1) &= A\,\underline{x}(k) + B\,\underline{u}(k) \\ \underline{y}(k) &= C\,\underline{x}(k) + D\,\underline{u}(k) . \end{aligned} \tag{4'}$$

<u>State-space equations for linear circuits</u>: To illustrate how state-space models arise, we describe a systematic procedure for obtaining state equations for linear circuits containing resistors, inductors and capacitors, voltage and current sources (possibly dependent) and op-amps operating in the linear region. The states that we select are the capacitor voltages and inductor currents. This is due to the fact that the constitutive relations

$$i = C\frac{dv}{dt} \qquad\qquad v = L\frac{di}{dt}$$

can be integrated as

$$v(t) - v(0) = \frac{1}{C}\int_0^t i(s)\,ds \qquad\qquad i(t) - i(0) = \frac{1}{L}\int_0^t v(s)\,ds . \tag{5}$$

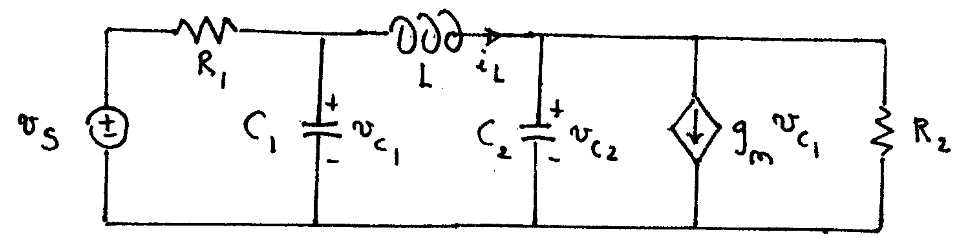
Thus given the initial capacitor voltage $v(0)$ (the capacitor "state") and future currents $i(s)$ $0 \le s \le t$, we can find the capacitor voltage $v(t)$ at $t > 0$. In other words $v(0)$ provides a complete summary of the past as far as describing the future behavior of the capacitor is concerned. Similarly, the initial inductor current $i(0)$, along with future voltages $v(s)$ for $0 \leq s \leq t$, is sufficient to determine the inductor current $i(t)$ for $t > 0$.

Another way of interpreting the concept of state consists in noting that the energy stored in a capacitor or inductor at time t is given by

$$E_c(t) = C\,\frac{v^2(t)}{2} \quad \text{or} \quad E_L(t) = L\,\frac{i^2(t)}{2}.$$

Thus specifying the capacitor voltage $v(t)$ or inductor current $i(t)$ is equivalent to specifying the energy stored in the capacitor or inductor at time t.

There exists a systematic procedure for obtaining state equations for a linear circuit. Consider the circuit



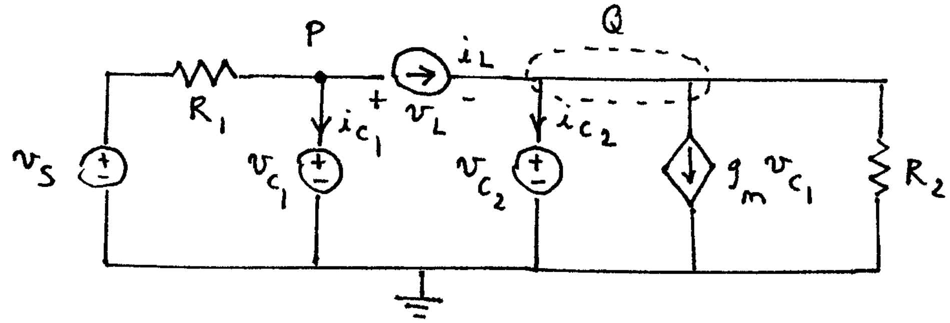
To obtain state equations for this circuit, we go through the following steps:

1) Replace temporarily each inductor L_j by a current source $i_{L_j}(t)$ and each capacitor C_k by a voltage source $v_{C_k}(t)$.

2) Solve the resulting circuit for the voltages $v_{L_j}(t)$ across the current sources replacing the inductors, and for the currents $i_{C_k}(t)$ through the voltage sources replacing the capacitors. Since the circuit consists only of resistors and sources, standard circuit analysis techniques, such as the current and voltage divider rules, superposition, Thevenin and Norton equivalents, ... can be employed.

3) Set $v_{L_j}(t) = L_j \frac{di_{L_j}}{dt}$, $i_{C_k}(t) = C_k \frac{dv_{C_k}}{dt}$ on the left of these equations to obtain a set of 1st order differential equations in terms of the state variables (inductor currents and capacitor voltages).

Here:



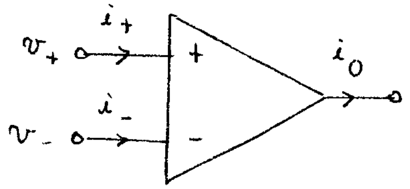
Applying the Kirchhoff voltage law (KVL) we find

$$v_L = v_{C_1} - v_{C_2}.$$

Similarly from the Kirchhoff current law (KCL) at points P and Q we get

at P: $i_{C_1} + i_L + \frac{v_{C_1} - v_S}{R_1} = 0 \Rightarrow i_{C_1} = -i_L + \frac{v_S - v_{C_1}}{R_1}$

op-amps. Recall that for an ideal op-amp, we have

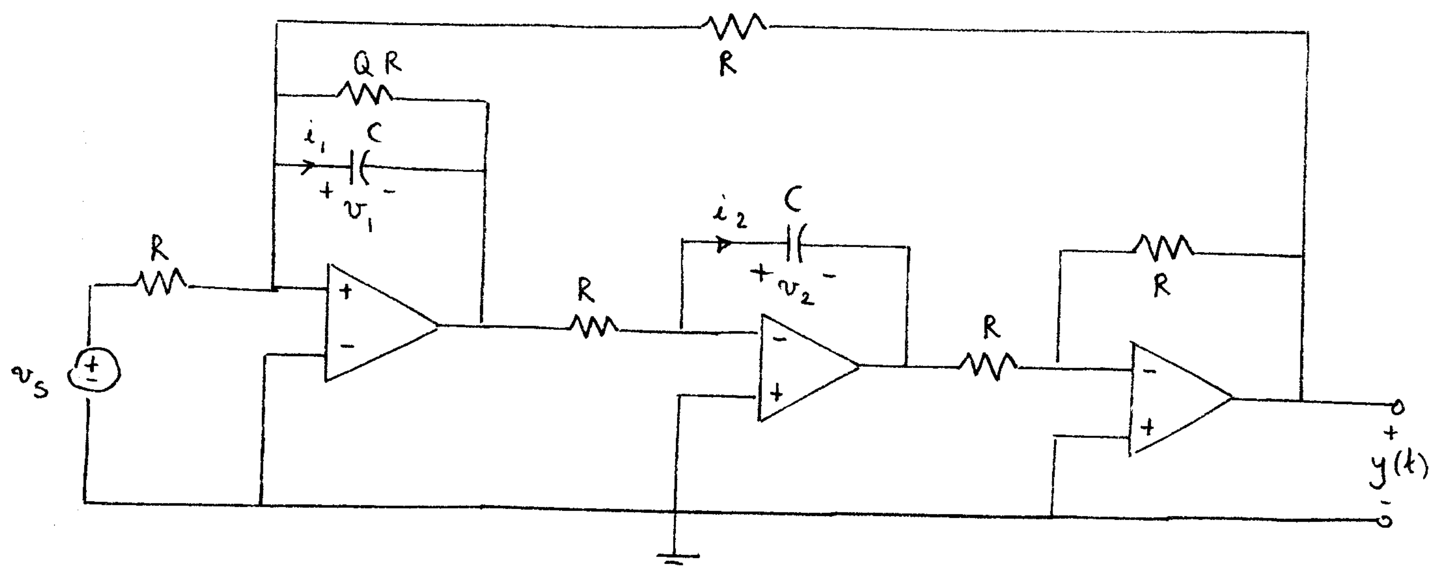


$$v_+ = v_-$$

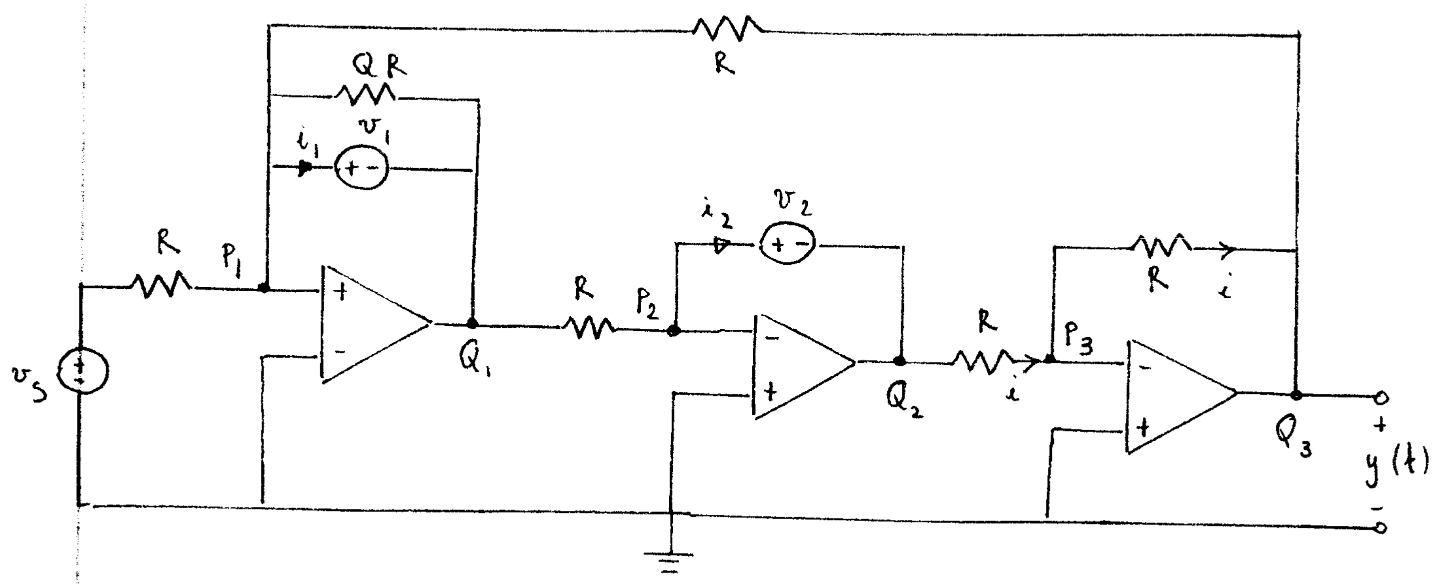
$$i_+ = i_- = 0 .$$

Note that this does not mean that the current i_o through the output terminal is zero, since the current is provided by two power supply terminals which are not shown here.

Consider the circuit



Replacing the two capacitors by voltage sources, we find



Since the op-amps are ideal, we have

$$v_{P_1} = v_{P_2} = v_{P_3} = 0$$

By inspection, we also see

$$v_{Q_1} = -v_1 \qquad v_{Q_2} = -v_2 \qquad v_{Q_3} = v_2 .$$

But the KCL at points P_1 and P_2 gives

at P_1: $\dfrac{v_1}{QR} + i_1 + \dfrac{(0-v_2)}{R} + \dfrac{(0-v_s)}{R} = 0 \Rightarrow \quad i_1 = -\dfrac{v_1}{QR} + \dfrac{v_2}{R} + \dfrac{v_s}{R}$

at P_2: $\quad i_2 = -\dfrac{v_1}{R}$.

Substituting $i_1 = C\dfrac{dv_1}{dt}$ and $i_2 = C\dfrac{dv_2}{dt}$, we find

$$\underbrace{\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\frac{d}{dt}x(t)} = \underbrace{\begin{bmatrix} -\frac{1}{CQR} & \frac{1}{CR} \\ -\frac{1}{CR} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{1}{R} \\ 0 \end{bmatrix}}_{B} v_s$$

$$y(t) = v_2(t) = \underbrace{[\,0 \quad 1\,]}_{C} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} .$$

<u>Realization of a scalar linear differential equation</u> : We now consider the problem of constructing a state-space model for the differential equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t)$$
$$= b_1 u^{(n-1)}(t) + \dots + b_n u(t) \qquad (6)$$

with n = order of the equation.

In the Laplace domain, the differential equation (6) takes the following form. We recall that the <u>unilateral Laplace transform</u> $Y(s)$ of a function $y(t)$ is defined as

$$Y(s) = \int_{0_-}^{\infty} y(t)\, e^{-st}\, dt .$$

Then an important property of the Laplace transform (LT) is that if $y(t) \longleftrightarrow Y(s)$ is a LT pair, then

$$\dot{y}(t) \longleftrightarrow sY(s) - y(0_-) \qquad (7)$$

is also a LT pair. More generally

$$y^{(k)}(t) \longleftrightarrow s^k Y(s) - \left(y^{(k-1)}(0_-) + s\, y^{(k-2)}(0_-) \dots + s^{k-1} y(0_-)\right) .$$

Then taking the LT of the differential equation (6) yields

$$a(s)\,Y(s) - y^{(n-1)}(0_-) - (s + a_1)\, y^{(n-2)}(0_-) \dots - (s^{n-1} + a_1 s^{n-2} \dots + a_{n-1})\, y(0_-)$$
$$= b(s)\,U(s) - b_1 u^{(n-2)}(0_-) \dots - (b_1 s^{n-2} \dots + b_{n-1})\, u(0_-) \qquad (8)$$

with

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_n$$

$$b(s) = b_1 s^{n-1} + \ldots + b_n \; .$$

To compute the transfer function $H(s)$ corresponding to the differential equation (6) we set all initial conditions equal to zero, i.e.

$$y(0_-) = \dot{y}(0_-) = \ldots = y^{(n-1)}(0_-) = 0$$

$$u(0_-) = \dot{u}(0_-) = \ldots = u^{(n-2)}(0_-) = 0 \; .$$

Then $$H(s) = \frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)} \qquad (9)$$

is a <u>rational</u> transfer function since it is the ratio of two polynomials $b(s)$ and $a(s)$. Furthermore the degree of $b(s)$, which is $n-1$ is strictly less than the degree of $a(s)$, which is n. This implies

$$\lim_{s \to \infty} H(s) = 0 \, , \qquad (10)$$

i.e. the transfer function $H(s)$ is <u>strictly proper</u>.

The realization of the differential equation (6) is performed in two steps:

1) We realize the LTI system $u(t) \xrightarrow{S_1} z(t)$ specified by the differential equation

$$z^{(n)}(t) + a_1 z^{(n-1)}(t) + \ldots + a_n z(t) = u(t) \, . \qquad (11)$$

Note that in the Laplace domain, the transfer function of S_1 is given by

$\frac{Z(s)}{U(s)} = \frac{1}{a(s)}$, i.e.

$$u(t) \longrightarrow \overset{S_1}{\boxed{\frac{1}{a(s)}}} \longrightarrow z(t)$$

2) Since S_1 is LTI, if $u(t) \xrightarrow{S_1} z(t)$ is an input/output pair, then

$$\dot{u}(t) \xrightarrow{S_1} \dot{z}(t)$$

$$u^{(k)}(t) \xrightarrow{S_1} z^{(k)}(t)$$

are also input/output pairs. By superposition, we have therefore

$$b_1 u^{(n-1)}(t) + \ldots + b_n u(t) \xrightarrow{S_1} b_1 z^{(n-1)}(t) + \ldots + b_n z(t)$$

But from the definition of S_1, if $y(t) = b_1 z^{(n-1)}(t) + \ldots + b_n z(t)$ is the output of S_1 corresponding to the input $v(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t)$, $y(t)$ satisfies the differential equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) \ldots + a_n y(t) = v(t)$$
$$= b_1 u^{(n-1)}(t) + \ldots + b_n u(t),$$

i.e. $y(t)$ satisfies the differential equation (6).

But the transfer function corresponding to the system $z(t) \xrightarrow{S_2} y(t)$ with $y(t) = b_1 z^{(n-1)}(t) + \ldots + b_n z(t)$ is $\frac{Y(s)}{Z(s)} = b(s)$, so that the LTI system S corresponding to (6) can be realized as

$$u(t) \longrightarrow \underbrace{\overset{S_1}{\boxed{\frac{1}{a(s)}}} \xrightarrow{z(t)} \overset{S_2}{\boxed{b(s)}}}_{S} \longrightarrow y(t)$$

Our implementation of the differential equation (6) relies on the following building blocks

integrators $$f(t) \longrightarrow \boxed{\int}^{\downarrow g(0)} \longrightarrow g(t) = g(0) + \int_0^t f(u)\,du$$

adders: $$\begin{array}{c} f_1(t) \\ \vdots \\ f_i(t) \\ \vdots \\ f_k(t) \end{array} \longrightarrow \oplus \longrightarrow g(t) = \sum_{i=1}^{k} f_i(t)$$

scalar multipliers: $f(t) \xrightarrow{a} g(t) = a\,f(t)$

Each of these building blocks can itself be implemented with op-amps, resis-tors and capacitors, but we will not be concerned here with the details of such implementations.

To implement S_1, we write the differential equation (11) as

$$z^{(n)}(t) = -a_1 z^{(n-1)}(t) \ldots - a_n z(t) + u(t) \tag{12}$$

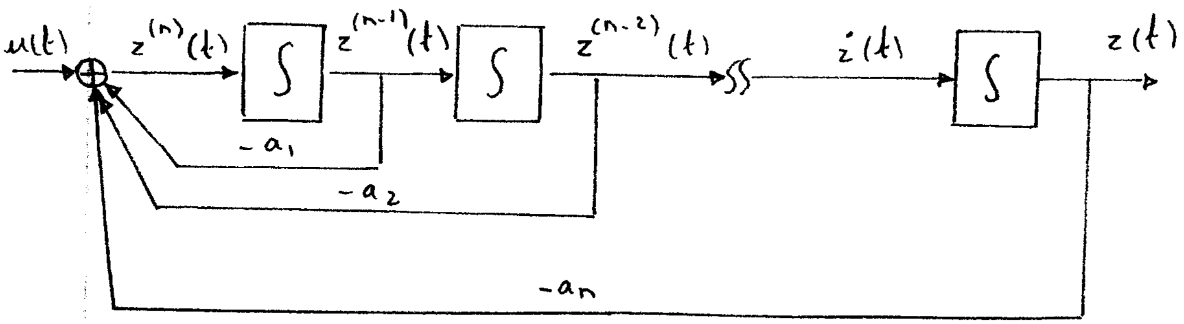
Consider now a chain of n integrators with input $z^{(n)}(t)$ and output $z(t)$, i.e.

$$z^{(n)}(t) \longrightarrow \boxed{\int} \xrightarrow{z^{(n-1)}(t)} \boxed{\int} \xrightarrow{z^{(n-2)}(t)} \cdots \cdots \xrightarrow{\dot z(t)} \boxed{\int} \longrightarrow z(t) \quad .$$

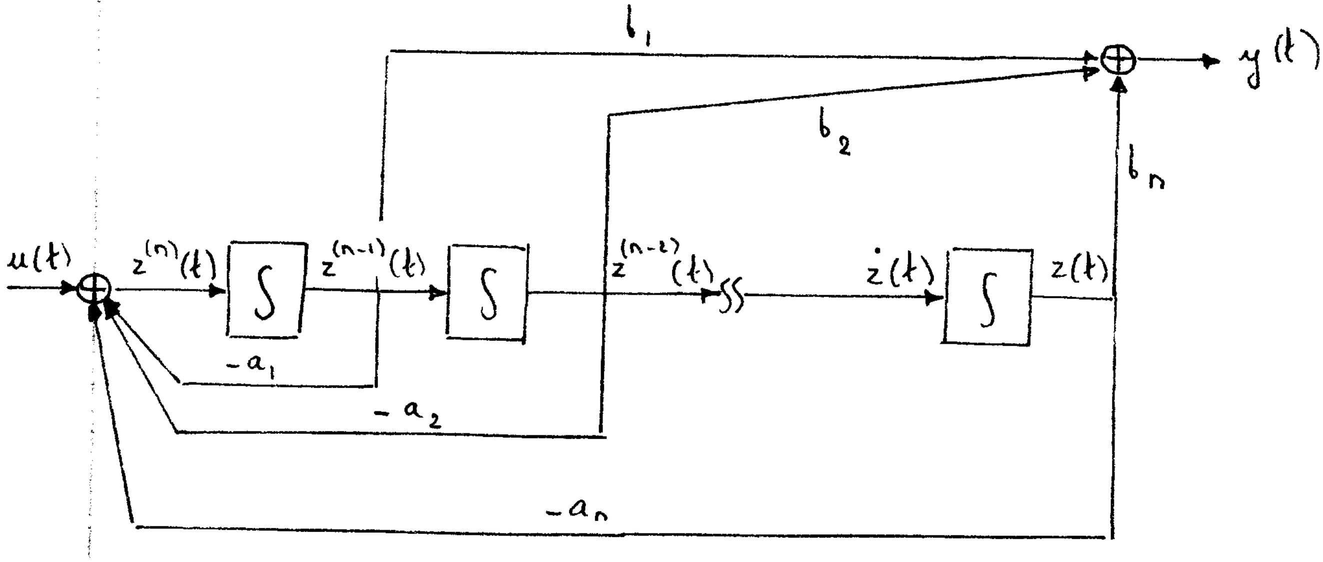
To implement (12) we need only to express the input $z^{(n)}(t)$ of the integrator chain as a linear combination of the input $u(t)$ and the outputs $z^{(n-1)}(t)$

... $z(t)$ of the integrators.

This gives



To implement the second step of the simulation procedure, we perform a linear combination $y(t) = b_1 z^{(n-1)}(t) + \ldots + b_n z(t)$ of the integrator outputs. The overall simulation is therefore given by



The above block diagram yields a state-space realization of the differential equation (6). To see this, we employ the <u>outputs</u> of the integrators as <u>states</u>. Proceeding from left to right, this gives

$$x_1(t) = z^{(n-1)}(t) \quad x_2(t) = z^{(n-2)}(t) \quad \ldots \quad x_n(t) = z(t). \tag{13}$$

The derivatives of the states are therefore the integrator inputs.
By inspection, we find

$$\dot{x}_1 = z^{(n)}(t) = -a_1 x_1(t) - a_2 x_2(t) \dots - a_n x_n(t) + u(t)$$

$$\dot{x}_2(t) = x_1(t)$$

$$\dots$$

$$\dot{x}_n(t) = x_{n-1}(t)$$

$$y(t) = b_1 x_1(t) + b_2 x_2(t) \dots + b_n x_n(t)$$

or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{b_c} u(t) \qquad (14a)$$

$$y(t) = \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}}_{c_c} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} . \qquad (14b)$$

The above state-space realization is called the controller canonical realization of $H(s) = \frac{b(s)}{a(s)}$. Note that the coefficients of the polynomials $a(s)$ and $b(s)$ appear respectively in the first row of A_c and in the matrix c_c, so that the realization (A_c, b_c, c_c) can be obtained <u>by inspection</u> from the knowledge of the transfer function $H(s)$.

<u>Example</u>: The output $y(t)$ of the above realization does not contain a feedthrough term $d\,u(t)$. This is due to the fact that the transfer function $H(s)$ was assumed to be strictly proper. Consider a case such as

$$\ddot{y} + 2\dot{y} - 2y = u - \dot{u} + \ddot{u}$$

where the order of differentiation of the input $u(t)$ is the same (two) as the order of differentiation of the output (two). In this case

$$H(s) = \frac{b(s)}{a(s)} = \frac{s^2 - s + 1}{s^2 + 2s - 2}$$

is proper, i.e. $\lim_{s\to\infty} H(s) = 1$ is finite, but not strictly proper.

Then we decompose $H(s)$ into a constant and a strictly proper component by performing the division

$$\underbrace{s^2 - s + 1}_{b(s)} = 1 \;\; \underbrace{(s^2 + 2s - 2)}_{a(s)} + \underbrace{(-3s + 3)}_{r(s)}$$

with degree of $r(s) = 1 <$ degree of $a(s) = 2$, so that

$$H(s) = \underbrace{\frac{-3s + 3}{s^2 + 2s - 2}}_{H_1(s)} + 1$$

Then the constant 1 can be implemented by the inclusion of a feedthrough term with $d = 1$ in the output, and $H_1(s)$ can be realized in controller form as indicated in (14). This gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) .$$