EEC 250 Linear Systems and Signals

Lecture 6

Topics: a) Signal flow graphs, Mason's rule

b) The four canonical realizations of differential / difference equations

c) Solution of state-space equations

Controller realization: In Lecture 5, it was shown that the transfer function of the system

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t) \qquad (1)$$

is $H(s) = \dfrac{b(s)}{a(s)}$ with

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_n$$

$$b(s) = \quad b_1 s^{n-1} + \ldots + b_n .$$

We also obtained a block diagram implementation of (1) in terms of integrators, adders and multipliers, which can be written in flow graph form as

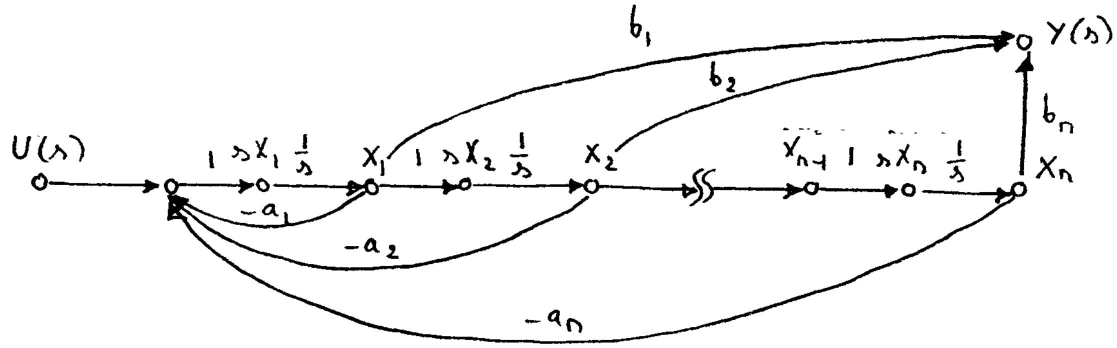


Figure 1

Here we have used the fact that an integrator can be represented in the Laplace domain by $\frac{1}{s}$. This is due to the fact that if $y(t) \leftrightarrow Y(s)$ is a Laplace transform (LT) pair, so is

$$\int_0^t y(u)\,du \leftrightarrow \frac{Y(s)}{s} .$$

The previous signal flow graph corresponds to the controller state-space realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{b_c} u(t) \qquad (2a)$$

$$y(t) = \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}}_{c_c} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} . \qquad (2b)$$

Although we have already verified that (2a)-(2b) is a state-space realization of (1), an independent confirmation consists in computing its transfer function. The Laplace transform of (2a)-(2b) is given by

$$s\,X(s) - \underline{x}(0_-) = A_c \underline{X}(s) + b_c U(s) \qquad (3a)$$

$$Y(s) = c_c \underline{X}(s) . \qquad (3b)$$

To compute the transfer function, we set the initial vector

$$\underline{x}(0_-) = \begin{bmatrix} x_1(0_-) \\ \vdots \\ x_n(0_-) \end{bmatrix} = \underline{0} \, .$$

Then

$$(sI_n - A_c)\underline{X}(s) = b_c\, U(s) \;\Rightarrow\; \underline{X}(s) = (sI_n - A_c)^{-1} b_c\, U(s)$$

so that $$Y(s) = c_c\,(sI_n - A_c)^{-1} b_c\, U(s) \, . \qquad (4)$$

It is easy to check that the determinant of

$$sI_n - A_c = \begin{bmatrix} s+a_1 & a_2 & \cdots & & a_n \\ -1 & s & & & \\ & \ddots & \ddots & & 0 \\ 0 & & \ddots & s & \\ & & & -1 & s \end{bmatrix}$$

is $\det(sI_n - A_c) = a(s) = s^n + a_1 s^{n-1} + \ldots + a_n$, and

$$(sI_n - A_c)^{-1} b_c = \frac{1}{a(s)} \begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix}$$

so that

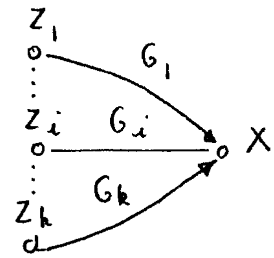
$$c_c\,(sI_n - A_c)^{-1} b_c = \frac{1}{a(s)} [\, b_1 \ldots b_n \,] \begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} = \frac{b(s)}{a(s)} = H(s) \, ,$$

as desired.

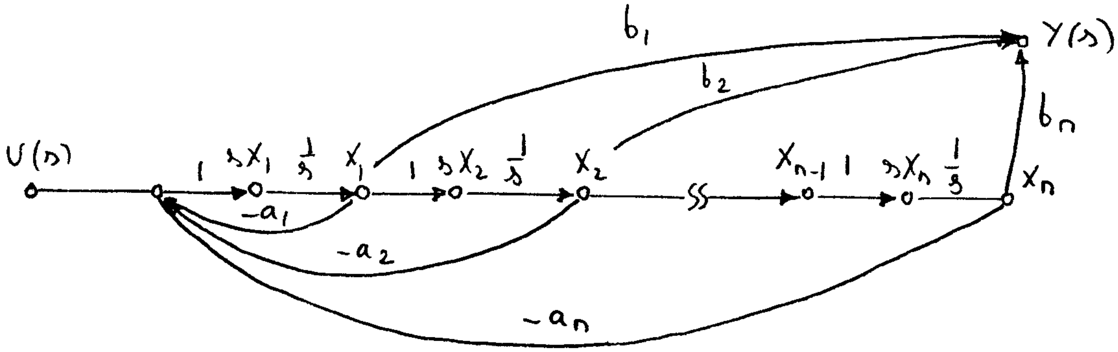
<u>Signal flow graphs</u>: A signal flow graph is just a pictorial way of representing a system of linear equations. It is constituted of nodes

and oriented branches. The nodes represent variables, and the branches are labeled by their gains. Consider an arbitrary node with variable X. If k branches with gains $G_1 \dots G_k$ enter this node, and the variables at the originating nodes are $Z_1 \dots Z_k$, X is given by

$$X = \sum_{i=1}^{k} G_i Z_i$$



<u>Example</u>: For the signal flow graph



we have

$$s X_1(s) = -a_1 X_1(s) - a_2 X_2(s) \dots - a_n X_n(s) + U(s)$$

$$s X_2(s) = X_1(s)$$

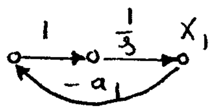
$$\dots$$

$$s X_n(s) = X_{n-1}(s)$$

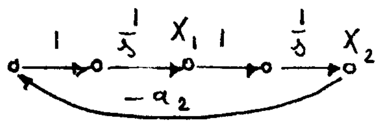
$$Y(s) = b_1 X_1(s) + b_2 X_2(s) + \dots + b_n X_n(s)$$

Elementary loop: An elementary loop of a signal flow graph is a closed path where no node is visited twice. The product L of the branch gains along an elementary loop is called the loop gain.

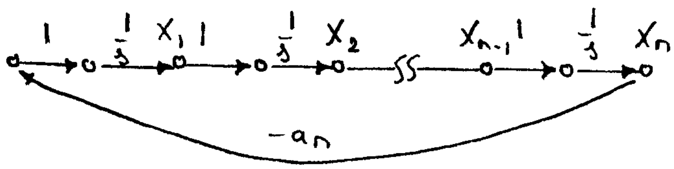
<u>Example</u>: The signal flow graph described above has n elementary loops



$L_1 = -a_1 s^{-1}$



$L_2 = -a_2 s^{-2}$

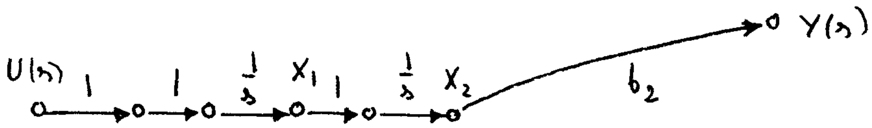


$L_n = -a_n s^{-n}$

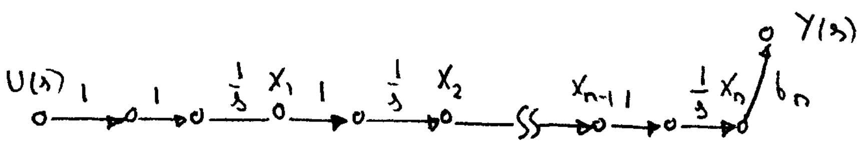
<u>Direct path</u>: A path linking two points in a signal flow graph is a direct path if no node is visited twice. The path gain P is the product of the branch gains along the path.

For the signal flow graph discussed earlier, the direct paths linking the input and output nodes U(s) and Y(s) are as follows:

$P_1 = b_1 s^{-1}$



$P_2 = b_2 s^{-2}$



$P_n = b_n s^{-n}$

<u>Determinant</u>: The determinant D of a signal flow graph is given by

$$D = 1 - \sum L_i + \sum L_i L_j - \sum L_i L_j L_k + \dots \qquad (5)$$

where in

(i) $\sum L_i$, the summation is over all elementary loops in the graph, with L_i = gain of the i^{th} loop

(ii) $\sum L_i L_j$, the summation is over all pairs of nontouching elementary loops in the graph. Two loops are said to be touching if they have at least a node in common.

(iii) $\sum L_i L_j L_k$, the summation is over all triples of mutually nontouching elementary loops (any pair of loops must be nontouching).

For the example considered above, two arbitrary elementary loops are necessarily touching, so that

$$D = 1 - \sum_{i=1}^{n} L_i = 1 + a_1 s^{-1} + a_2 s^{-2} + \dots + a_n s^{-n}$$

<u>Cofactor</u>: Given a direct path P_i linking two nodes in the signal flow graph, the cofactor D_i of P_i is the determinant of the signal flow graph obtained by removing all nodes belonging to the path P_i, and all branches originating from or arriving to these nodes.

For the example considered here, the cofactors of the direct paths $P_1, \ldots P_n$ linking $U(s)$ and $Y(s)$ are all 1, since as soon as we remove the branch $\overset{sX_1}{\circ}\!\longrightarrow\!\overset{X_1}{\circ}$ and the corresponding nodes, all elementary loops of the signal flow graph disappear, i.e. $D_1 = D_2 = \ldots = D_n = 1$.

<u>Mason's rule</u>: Given a signal flow graph, the transfer function from an input node to an output node is given by

$$H = \frac{\sum P_i D_i}{D} \tag{6}$$

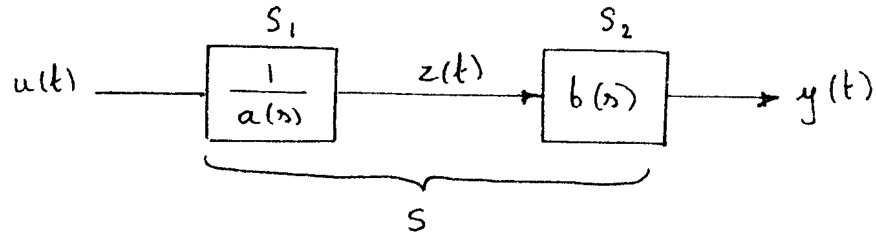
where D = determinant of the flow graph an in $\sum P_i D_i$, the summation is over all direct paths linking the input node to the output node with $P_i = i^{th}$ path gain and $D_i = i^{th}$ path cofactor.

For the flow graph corresponding to the controller canonical realization, we have $D = 1 + \sum_{i=1}^{n} a_i s^{-i}$, $P_i = b_i s^{-i}$ and $D_i = 1$, so that

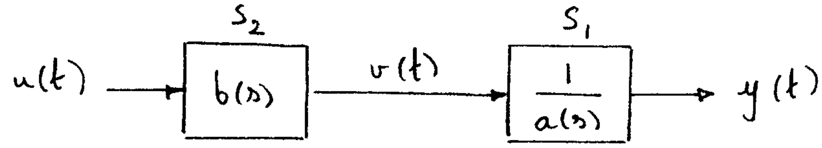
$$\frac{Y(s)}{U(s)} = H(s) = \frac{\sum_{i=1}^{n} b_i s^{-i}}{1 + \sum_{i=1}^{n} a_i s^{-i}} = \frac{b(s)}{a(s)}$$

as was already shown.

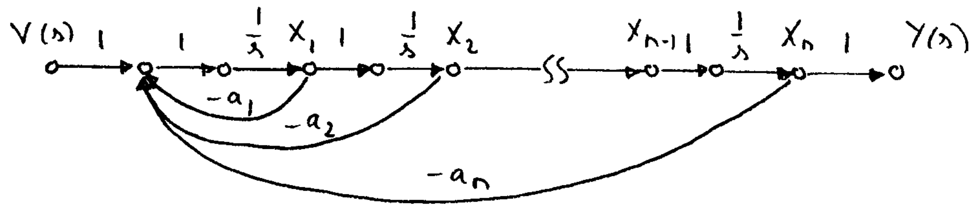
<u>Observability canonical form</u>: To derive the controller form, we expressed the system S with transfer function $\frac{b(s)}{a(s)}$ as the cascade of two systems S_1 and S_2 with transfer functions $\frac{1}{a(s)}$ and $b(s)$, respectively



Since S_1 and S_2 are linear time-invariant (LTI), they can be inter-changed, thus yielding a realization of the form



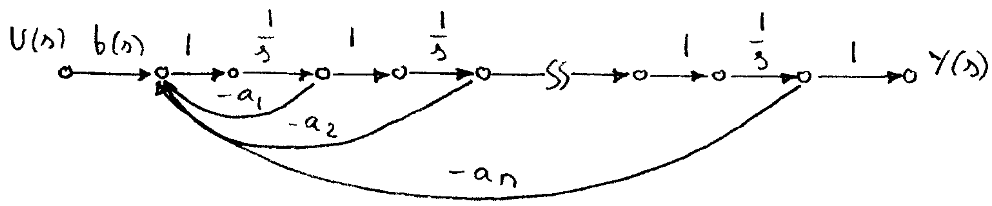
It was shown earlier that a signal flow graph realization of S_1 is given by



On the other hand since $b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n$, the output $v(t)$ of S_2 can be expressed in terms of its input $u(t)$ as

$$v(t) = b_1 u^{(n-1)}(t) + b_2 u^{(n-2)}(t) + \ldots + b_n u(t),$$

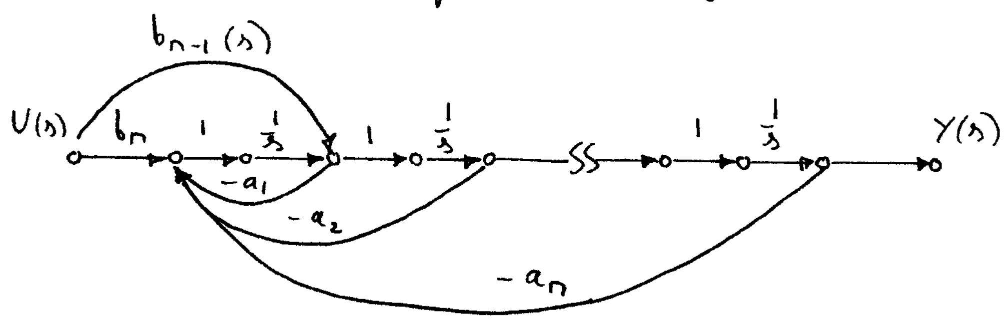
i.e. the implementation of S_2 requires the use of differentiators. However the flow graph implementation of S_1 shown above contains a chain of n integrators that can be used to remove the differentiation operations required to produce $v(t)$. We start from the flow graph



But $b(s)$ can be decomposed as

$$b(s) = s\,\underbrace{(b_1 s^{n-2} + \ldots + b_{n-1})}_{b_{n-1}(s)} + b_n$$

Since the differentiation corresponding to the s multiplication is immediately followed by an integration, it can be removed by injecting $b_{n-1}(s)$ at the output of the 1st integrator, thus yielding



But $b_{n-1}(s)$ can itself be decomposed as

$$b_{n-1}(s) = s\,(b_1 s^{n-3} + \ldots + b_{n-2}) + b_{n-1}$$

where the s factor can again be removed by integration. The problem in doing so is that we are altering the signal transmitted by the $-a_1$ feedback path. The coefficients b_i $1 \leq i \leq n$ must be modified appropriately. To see how this can be achieved, consider the signal flow graph

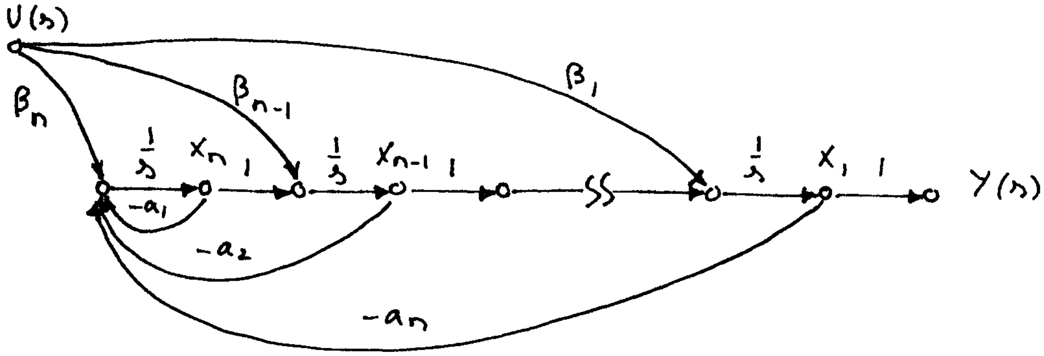
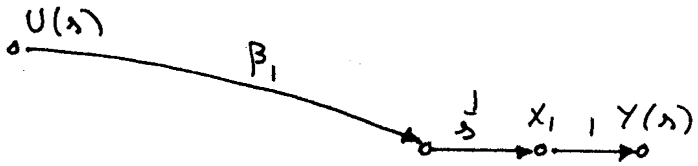


Figure 2

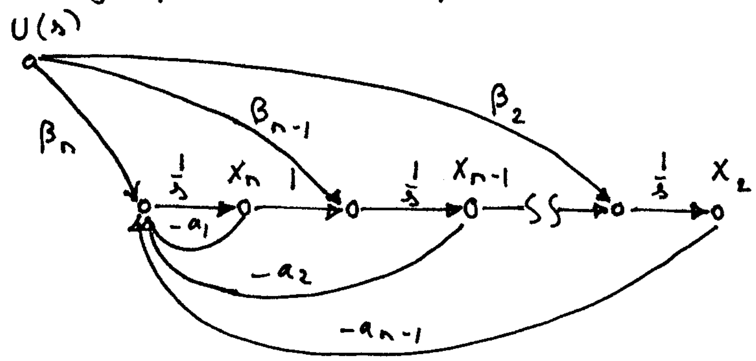
The elementary loops for this signal flow graph are the same as for the controller form, so that $D = 1 + \sum_{i=1}^{n} a_i s^{-i}$. On the other hand, the direct paths between the input and output nodes U(s) and Y(s), and their cofactors are as follows.

<u>Path # 1</u>:



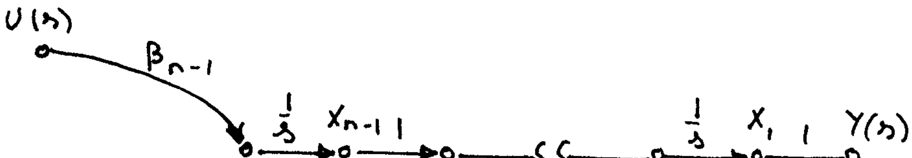
$P_1 = \beta_1 s^{-1}$

The subgraph with this path removed is



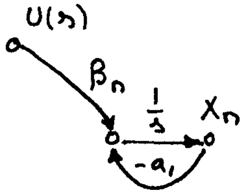
The n^{th} elementary loop has been removed, so that the cofactor $D_1 = 1 + \sum_{i=1}^{n-1} a_i s^{-i}$

<u>Path # n-1</u>:



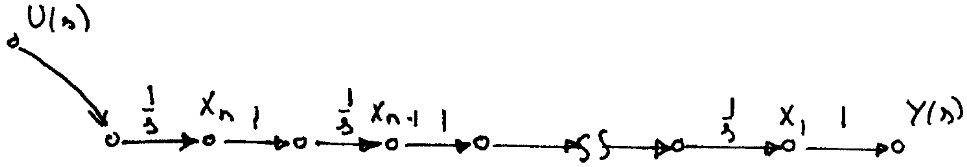
$P_{n-1} = \beta_{n-1} s^{-(n-1)}$

The subgraph associated to its cofactor is



$$D_{n-1} = 1 + a_1 s^{-1}$$

<u>Path # n</u>:



$$P_n = \beta_n s^{-n}$$

The subgraph obtained by removing this path has no elementary loop, so that $D_n = 1$.

The transfer function for the observability form of Figure 2 is therefore given by

$$H(s) = \frac{\sum_{i=1}^{n} P_i D_i}{D}$$

with

$$\sum_{i=1}^{n} P_i D_i = \beta_1 s^{-1}\left(1 + \sum_{i=1}^{n-1} a_i s^{-i}\right) + \ldots + \beta_{n-1} s^{-(n-1)} (1 + a_1 s^{-1}) + \beta_n s^{-n}$$

$$= \beta_1 s^{-1} + (\beta_2 + a_1 \beta_1) s^{-2} + \ldots + (\beta_n + a_1 \beta_{n-1} + \ldots + a_{n-1} \beta_1) s^{-n} \quad (7)$$

$$D = 1 + \sum_{i=1}^{n} a_i s^{-i} . \quad (8)$$

Equating this transfer function to $b(s)/a(s)$, we see that (7) must be equal to

$$s^{-n} b(s) = b_1 s^{-1} + b_2 s^{-2} + \ldots + b_n s^{-n} \quad (9)$$

Identifying coefficients of $s^{-1}, s^{-2}, \ldots s^{-n}$ in (7) and (9) gives

$$b_1 = \beta_1$$

$$b_2 = \beta_2 + a_1 \beta_1$$

$$\cdots$$

$$b_n = \beta_n + a_1 \beta_{n-1} + \cdots + a_{n-1} \beta_1$$

or equivalently in matrix form

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & 0 \\ a_1 & 1 & & \\ & \ddots & \ddots & \\ a_{n-1} & & a_1 & 1 \end{bmatrix}}_{M} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \qquad (10)$$

where the matrix M is constant along its diagonals, i.e. it has 1 along its diagonal, a_1 along the first subdiagonal...

Thus, the coefficients β_i $1 \leq i \leq n$ of the observability realization of Figure 2 can be expressed in terms of the coefficients a_j, b_j with $1 \leq j \leq n$ of the transfer function $H(s) = b(s)/a(s)$ through

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \\ \beta_n \end{bmatrix} = M^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} .$$

The observability canonical realization of Figure 2 can be written in state-space form by using the output nodes of the $\frac{1}{s}$ integrators as states, which are numbered from right to left. This gives

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & & & 0 \\ \vdots & & \ddots & & \\ 0 & \cdots & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}}_{A_{ob}} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}}_{b_{ob}} u(t) \qquad (11a)$$

$$y(t) = \underbrace{[1 \ 0 \ \ldots \ 0]}_{c_{ob}} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad (11b)$$

This could also be seen by rewriting the signal flow graph of Figure 2 in block diagram form as

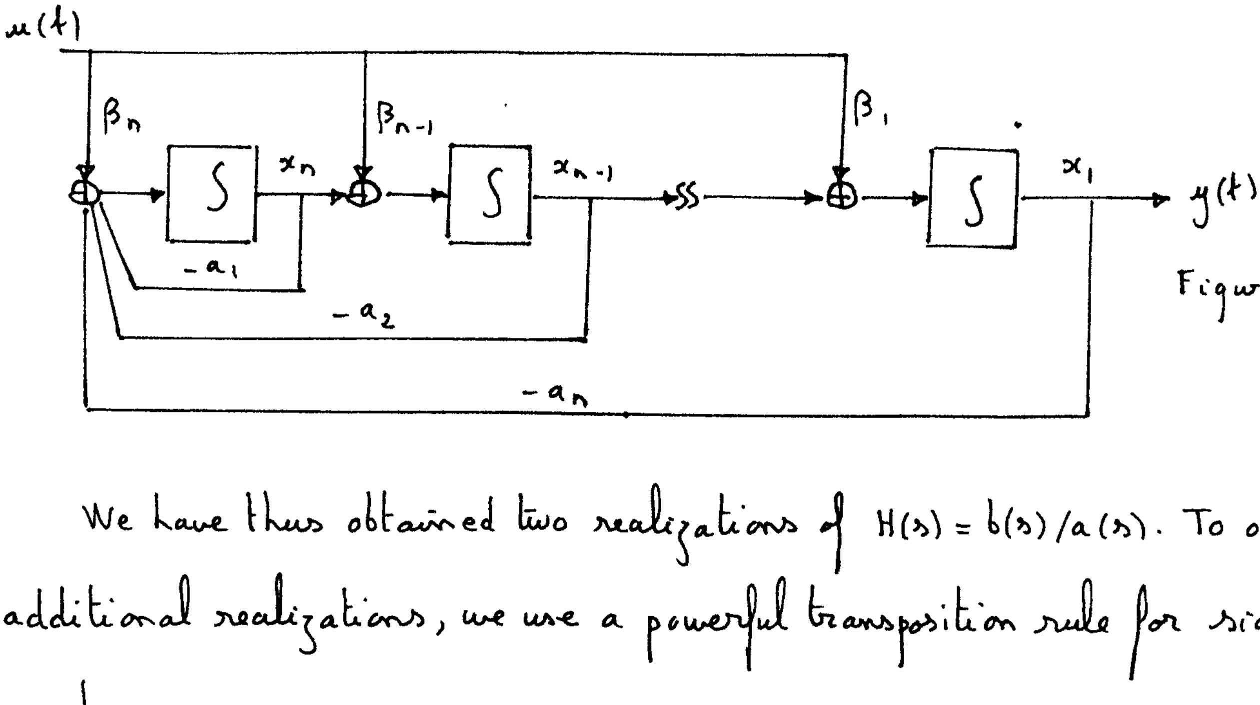


Figure 3

We have thus obtained two realizations of $H(s) = b(s)/a(s)$. To obtain two additional realizations, we use a powerful transposition rule for signal flow graphs.

<u>Transposition rule</u>: If $H(s)$ is the transfer function corresponding to a signal flow graph, $H^T(s)$ is the transfer function of the signal flow graph

obtained by replacing input nodes by output nodes, and vice-versa, and reversing the direction of all branches in the graph while keeping th branch gains as they were.

The validity of the above transposition rule is a consequence of Mason's rule. Since transposition changes only the direction of all elementary loops and direct input/output paths, the input/output transfer functions are preserved.

In the scalar case, i.e. for a graph with a single input node and a single output node, $H^T(s) = H(s)$. Thus, applying the transposition rule to the signal flow graphs representing the controller and observability realizations of $H(s) = b(s)/a(s)$ yields two new realizations of $H(s)$: the observer and controllability realizations.

<u>Observer realization</u>: Applying the transposition rule to the signal flow graph of Figure 1 yields

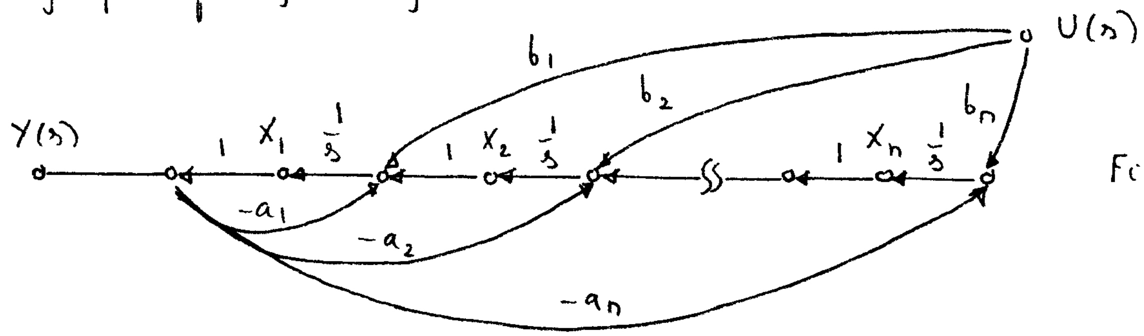


Figure 4

This signal flow graph can be transformed to block diagram form, which for convenience is drawn from left to right

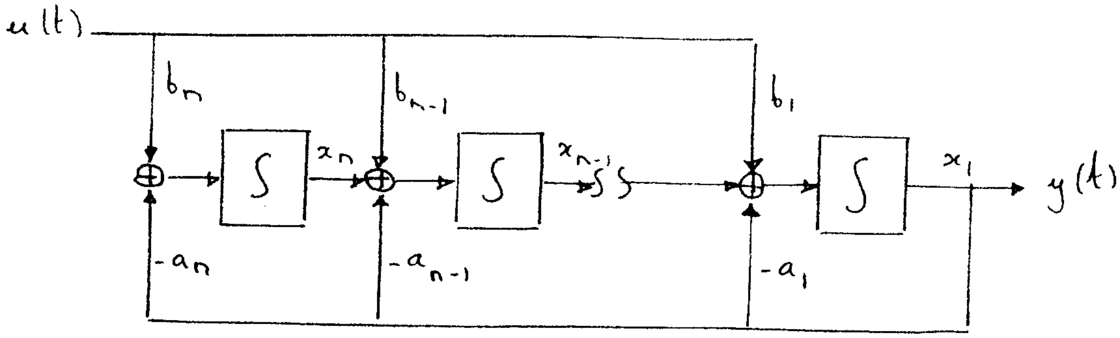


Figure 5

The state-space equations corresponding to this realization are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & 1 & & & 0 \\ -a_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ -a_{n-1} & \vdots & & & 1 \\ -a_n & 0 & \cdots & \cdots & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}}_{b_0} u(t) \qquad (12a)$$

$$y(t) = \underbrace{[\,1 \quad 0 \; \cdots \; 0\,]}_{c_0} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (12b)$$

Note that the matrices (A_0, b_0, c_0) can be expressed in terms of the matrices (A_c, b_c, c_c) of the controller realization as

$$A_0 = A_c^T, \quad b_0 = c_c^T \quad c_0 = b_c^T \qquad (13)$$

These relations are just a consequence of the transposition rule.

Controllability realization: Similarly, applying the transposition rule to the signal flow graph of Figure 2 yields

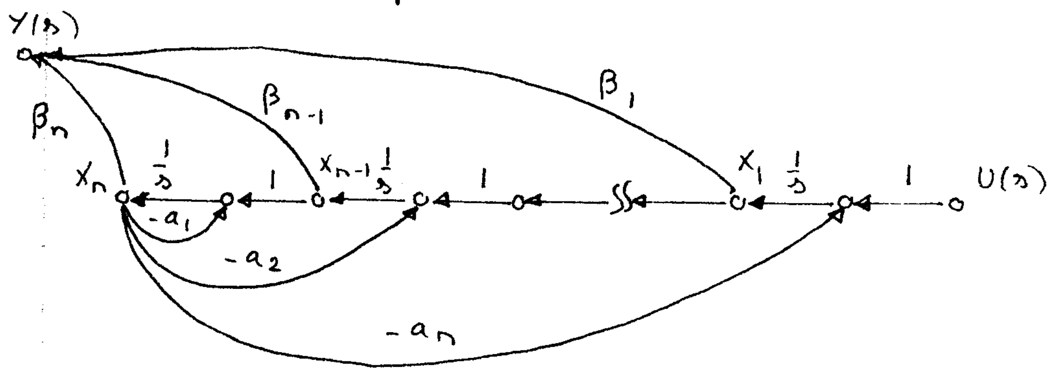


Figure 6

which can be rewritten from left to right in block diagram form as

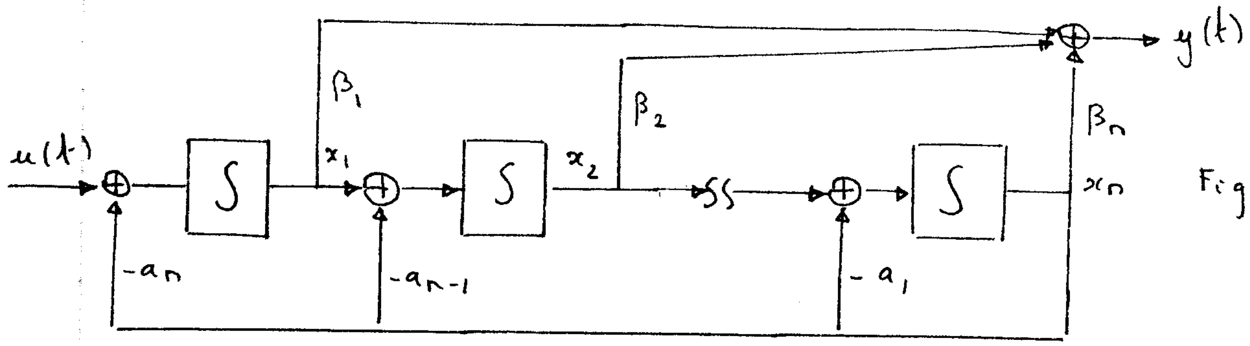


Figure 7

The state-space equations for this realization are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_n \\ 1 & 0 & & \vdots & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ 0 & & & 1 & -a_1 \end{bmatrix}}_{A_{co}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{b_{co}} u(t) \qquad (14a)$$

$$y(t) = \underbrace{[\beta_1 \;\; \beta_2 \;\; \cdots \;\; \beta_n]}_{c_{co}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (14b)$$

The matrices (A_{co}, b_{co}, c_{co}) are related to the matrices (A_{ob}, b_{ob}, c_{ob}) of the observability realization through the identities

$$A_{co} = A_{ob}^{T} \quad , \quad b_{co} = c_{ob}^{T} \qquad c_{co} = b_{ob}^{T} \tag{15}$$

which are again a consequence of the transposition rule used to obtain the controllability realization from the observability form.

<u>Difference equations</u>: Up to this point we have focused on the realization of differential equations. Suppose that we consider instead the difference equation

$$y(k+n) + a_1 y(k+n-1) + \ldots + a_n y(k) = b_1 u(k+n-1) + \ldots + b_n u(k). \tag{16}$$

To compute its transfer function we employ the unilateral z-transform

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} \quad .$$

Two key properties of the z-transform are as follows: if $f(k) \leftrightarrow F(z)$ is a z-transform pair, then

$$f(k+1) \leftrightarrow z\,(F(z) - f(0)) \tag{17a}$$

$$f(k-1) \leftrightarrow z^{-1} F(z) \quad . \tag{17b}$$

In other words the time-delay operation $Df(k) = f(k-1)$ is similar to the integration operation, since it is represented by a multiplication by z^{-1} in the z-domain, while integration was represented by s^{-1} in the s domain.

Then, by taking the z-transform of (16), we find

$$a(z)Y(z) - z\,y(n-1) - (z^2 + a_1 z)\,y(n-2) \ldots - (z^n + a_1 z^{n-1} + \ldots + a_{n-1} z)\,y(0)$$
$$= b(z)\,Y(z) - b_1 z\,u(n-1) \ldots - (b_1 z^{n-1} \ldots + b_{n-1} z)\,u(0). \qquad (18)$$

To obtain the transfer function corresponding to the difference equation (16), we set all initial conditions equal to zero, i.e.

$$y(0) = y(1) = \ldots = y(n-1) = 0$$
$$u(0) = u(1) = \ldots = u(n-2) = 0 \quad .$$

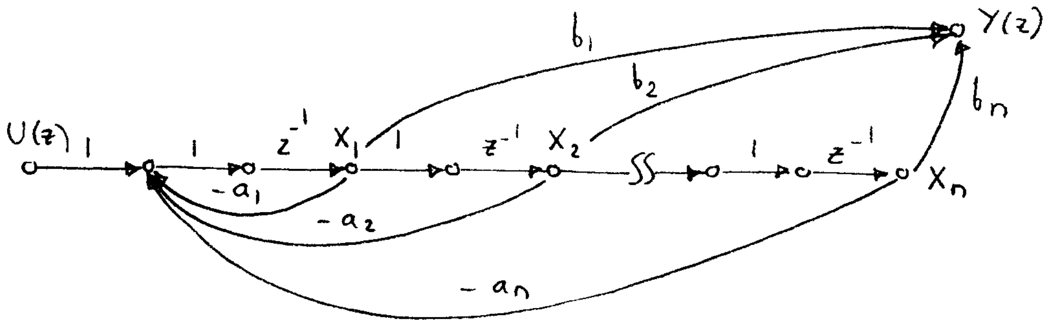
This gives $\frac{Y(z)}{U(z)} = H(z) = \frac{b(z)}{a(z)}$. Thus the transfer function of (16) is identical to the CT differential equation considered earlier, provided we replace s by z.

To realize $H(z)$, we employ block diagrams with adders, scalar multipliers and delay elements, i.e.

$$f(k) \longrightarrow \boxed{D} \longrightarrow f(k-1)$$

Since the delay elements and integrators are represented respectively by z^{-1} and s^{-1} in the z and s domains, any CT realization of $H(s)$ will yield a corresponding DT realization of $H(z)$, provided we replace all integrators by delay elements in block diagrams, or s^{-1} by z^{-1} in signal flow graphs.

For example, the signal flow graph of Figure 1 is transformed into



and the corresponding state-space equations are given by

$$\underline{x}(k+1) = A_c\,\underline{x}(k) + b_c\,u(k)$$

$$y(k) = c_c\,\underline{x}(k)$$

where the matrices (A_c, b_c, c_c) are as in (2a)-(2b).

Solution of state-space equations: Consider the state-space model

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{19a}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{19b}$$

with $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\underline{y} \in \mathbb{R}^p$. Taking Laplace transforms we find

$$s\,\underline{X}(s) - \underline{x}(0) = A\,\underline{X}(s) + B\,\underline{U}(s) \tag{20a}$$

$$\underline{Y}(s) = C\,\underline{X}(s) + D\,\underline{U}(s) \tag{20b}$$

Thus $$\underline{X}(s) = (sI-A)^{-1}\underline{x}(0) + (sI-A)^{-1}B\,\underline{U}(s) \tag{21a}$$

$$\underline{Y}(s) = C(sI-A)^{-1}\underline{x}(0) + [C(sI-A)^{-1}B + D]\,\underline{U}(s) \tag{21b}$$

The transfer function of the system is obtained by setting all initial conditions $\underline{x}(0) \equiv 0$, which yields $\underline{Y}(s) = H(s)\,\underline{U}(s)$ with

$$H(s) = C(sI-A)^{-1}B + D \quad : \; p \times m \text{ transfer matrix}$$

To obtain time-domain expressions, we introduce

$$e^{At} = \text{matrix exponential} \triangleq LT^{-1}[(sI-A)^{-1}]$$

Then, the inverse Laplace transform of (21a) is

$$\underline{x}(t) = e^{At}\underline{x}(0) + e^{At}B * u(t) \tag{22}$$

where $*$ denotes the convolution operation, i.e.

$$e^{At}B * u(t) = \int_0^t e^{A(t-s)} B\, u(s)\, ds \,. \tag{23}$$

The decomposition (22) admits the following interpretation:

(i) $e^{At}\underline{x}(0)$ is the zero-input response (ZIR) of the system, since $\underline{x}(t) = e^{At}\underline{x}(0)$ whenever the input vector $\underline{u}(t) \equiv 0$ for all t.

(ii) $e^{At}B * u(t)$ is the zero-state response (ZSR) since $\underline{x}(t) = e^{At}B * u(t)$ whenever the initial state vector $\underline{x}(0) = 0$.

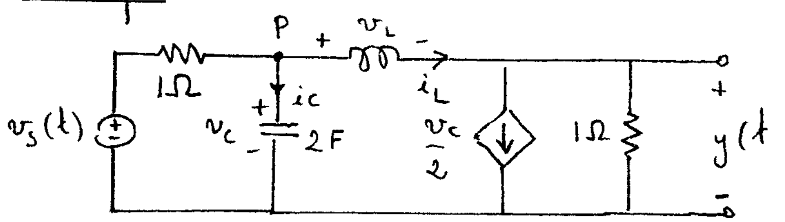
The output is then given by

$$\underline{y}(t) = Ce^{At}\underline{x}(0) + [Ce^{At}B + D\delta(t)] * \underline{u}(t) \tag{24}$$

so that we see that the matrix impulse response of the system is

$$H(t) = Ce^{At}B + D\delta(t)\,. \tag{25}$$

<u>Example</u>: Consider the circuit



with initial conditions

$v_C(0-) = -2\,V$, $i_L(0-) = 1\,A$.

The KCL at node P yields

$$i_C + i_L + v_C - v_S = 0 \Rightarrow i_C = -v_C - i_L + v_S$$

Also, from KVL we find

$$v_C = v_L + \left(i_L - \frac{v_C}{2}\right) \Rightarrow v_L = \frac{3}{2} v_C - i_L \ .$$

Substituting the constitutive relations $i_C = 2\frac{dv_C}{dt}$ $v_L = 2\frac{di_L}{dt}$ on the left-hand sides of the above identities yields

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & -\frac{1}{2} \end{bmatrix}}_{A} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{B} v_S(t)$$

$$\begin{aligned} y(t) &= i_L - \frac{v_C}{2} \\ &= \underbrace{\begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}}_{C} \begin{bmatrix} v_C \\ i_L \end{bmatrix} . \end{aligned}$$

But

$$sI - A = \begin{bmatrix} s+\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{4} & s+\frac{1}{2} \end{bmatrix} , \quad (sI-A)^{-1} = \begin{bmatrix} s+\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & s+\frac{1}{2} \end{bmatrix} \Big/ \left(s+\frac{1}{2}\right)^2 + \frac{3}{8} \ .$$

According to the Laplace transform tables

$$\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{8}} \longleftrightarrow e^{-t/2} \cos\left(\frac{1}{2}\sqrt{\frac{3}{2}}\, t\right)$$

$$\frac{\frac{1}{2}\sqrt{\frac{3}{2}}}{(s+\frac{1}{2})^2 + \frac{3}{8}} \longleftrightarrow e^{-t/2} \sin\left(\frac{1}{2}\sqrt{\frac{3}{2}}\, t\right)$$

so that

$$e^{At} = e^{-t/2}\begin{bmatrix} \cos(\frac{1}{2}\sqrt{\frac{3}{2}}t) & -\sqrt{\frac{2}{3}}\sin(\frac{1}{2}\sqrt{\frac{3}{2}}t) \\ \sqrt{\frac{3}{2}}\sin(\frac{1}{2}\sqrt{\frac{3}{2}}t) & \cos(\frac{1}{2}\sqrt{\frac{3}{2}}t)\end{bmatrix}$$

and the impulse response for this circuit is given by

$$H(t) = C e^{At} B = \frac{e^{-t/2}}{2}\left(-\frac{1}{2}\cos(\tfrac{1}{2}\sqrt{\tfrac{3}{2}}t) + \sqrt{\tfrac{3}{2}}\sin(\tfrac{1}{2}\sqrt{\tfrac{3}{2}}t)\right). \blacksquare$$

<u>Discrete-time case</u>: Similarly the z-transform of the DT model

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \tag{26a}$$

$$\underline{y}(k) = C\,\underline{x}(k) + D\,\underline{u}(k) \tag{26b}$$

takes the form

$$z(\underline{X}(z) - \underline{x}(0)) = A\,\underline{X}(z) + B\,\underline{U}(z)$$

$$\underline{Y}(z) = C\,\underline{X}(z) + D\,\underline{U}(z)$$

so that
$$\underline{X}(z) = (zI-A)^{-1} z\,\underline{x}(0) + (zI-A)^{-1}B\,\underline{U}(z) \tag{27a}$$

$$\underline{Y}(z) = C(zI-A)^{-1} z\,\underline{x}(0) + [C(zI-A)^{-1}B + D]\,\underline{U}(z), \tag{27b}$$

where $H(z) = C(zI-A)^{-1}B + D$ = transfer function matrix. Noting now that

$$(zI-A)^{-1} = z^{-1}(I-Az^{-1})^{-1} = \sum_{k=1}^{\infty} A^{k-1} z^{-k},$$

i.e.

$$(zI-A)^{-1} \leftrightarrow A^{k-1}1(k-1) = \begin{cases} A^{k-1} & \text{for } k \geq 1 \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

the solution of (26a) is given by

$$\underline{x}(k) = A^k \underline{x}(0) + A^{k-1} B\, 1(k-1) * u(k) \tag{29}$$

with

$$A^{k-1} B\, 1(k-1) * u(k) = \sum_{\ell=1}^{k-1} A^{k-\ell-1} B\, u(\ell) \; .$$

The output is given by

$$\underline{y}(k) = C A^k \underline{x}(0) + [C A^{k-1} B\, 1(k-1) + D\, \delta(k)] * u(k) \tag{30}$$

where

$$H(k) = C A^{k-1} B\, 1(k-1) + D\, \delta(k) = \begin{cases} 0 & \text{for } k < 0 \\ D & \text{for } k = 0 \\ C A^{k-1} B & \text{for } k \geq 1 \end{cases} \tag{31}$$

is the matrix impulse response of the system.