EEC 250 Linear Systems and Signals

Lecture 9

Topics: a) DT reachability

b) Reachability tests

c) Decomposition of a system into reachable/unreachable parts

Reachability: Consider the DT system

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \tag{1}$$

where the initial state is assumed to be zero, i.e. $\underline{x}(0) = \underline{0}$. The input vectors $\underline{u}(k)$ can be selected so that various state configurations $\underline{v}$ can be reached. This leads to the question: what are the states that can be reached in k steps?

For $k=1$, we have

$$\underline{x}(1) = A\,\underline{x}(0) + B\,\underline{u}(0) = B\,\underline{u}(0)$$

so that the states that can be reached in one step must be in the column space of B. For $k=2$

$$\underline{x}(2) = A\,\underline{x}(1) + B\,\underline{u}(1) = AB\,\underline{u}(0) + B\,\underline{u}(1)$$

$$= [B \quad AB] \begin{bmatrix} \underline{u}(1) \\ \underline{u}(0) \end{bmatrix}$$

so that the states that can be reached in two steps must be in the column space of $[B \ AB]$.

More generally, for k steps

$$x(k) = \sum_{l=0}^{k-1} A^{k-l-1} B\, u(l)$$

$$= [B \ AB \ \ldots \ A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix} \qquad (2)$$

so that the states that can be reached in k steps must be in the column space $\mathcal{R}_k$ of the k-step reachability matrix

$$R_k = [B \ AB \ \ldots \ A^{k-1}B]. \qquad (3)$$

The reachability spaces $\mathcal{R}_k$ have some important properties that we now investigate.

<u>Properties of</u> $\mathcal{R}_k$: 1) $\mathcal{R}_k \subseteq \mathcal{R}_{k+1}$. This property expresses the intuitive idea that we can reach more states in k+1 steps than in k steps. To prove it, we must show that if $\underline{v}$ is reached in k steps, it can be reached in k+1 steps. Let $\underline{u}(0) \ldots \underline{u}(k-1)$ be the sequence of inputs that takes the state vector from $\underline{x}(0) = \underline{0}$ to $\underline{x}(k) = \underline{v}$.

Then to reach $\underline{v}$ in $k+1$ steps all we need to do is apply no input for one step, so that $\underline{x}(1) = \underline{0}$, and then apply the input sequence $\underline{u}(0) \ldots \underline{u}(k-1)$ that we used earlier to reach $\underline{v}$ in k steps. Specifically if we select the input sequence

$$\underline{w}(0) = 0 \quad , \underline{w}(l) = \underline{u}(l-1) \quad \text{for } 1 \leq l \leq k$$

we have $\underline{x}(k+1) = \underline{v}$.

2) If $\mathcal{R}_{k+1} = \mathcal{R}_k$, then $\mathcal{R}_s = \mathcal{R}_k$ for all $s \geq k$. This property indicates that if the reachability space $\mathcal{R}_k$ stops growing as we go from integer k to $k+1$, it will not grow any longer.

<u>Proof</u>: If $\mathcal{R}_{k+1} = \mathcal{R}_k$, the column spaces of $R_{k+1} = [B \; AB \; \ldots \; A^k B]$ and $R_k = [B \; AB \; \ldots \; A^{k-1} B]$ are identical, so that the column vectors of $A^k B$ can be expressed as linear combinations of the column vectors of $B, AB \ldots A^{k-1}B$. This means there exists some $m \times m$ matrices $M_0 \ldots M_{k-1}$ such that

$$A^k B = [B \; AB \; \ldots \; A^{k-1} B] \begin{bmatrix} M_{k-1} \\ M_{k-2} \\ \vdots \\ M_0 \end{bmatrix} . \qquad (4)$$

Premultiplying (4) by A yields

$$A^{k+1}B = [AB \;\; A^2B \;\; \ldots \;\; A^kB] \begin{bmatrix} M_{k-1} \\ M_{k-2} \\ \vdots \\ M_0 \end{bmatrix} \tag{5}$$

This implies that the columns of $A^{k+1}B$ are a linear combination of the columns of $B, AB \ldots A^kB$ of $\mathcal{R}_{k+1}$, so that $\mathcal{R}_{k+2} = \mathcal{R}_{k+1} = \mathcal{R}_k$. By induction, we can deduce that $\mathcal{R}_s = \mathcal{R}_k$ for all $s \geq k$.

3) If A is an $n \times n$ matrix, $\mathcal{R}_{n+1} = \mathcal{R}_n$. Combining this property with the first two, this means that if a state $\underline{v}$ cannot be reached in n steps, it can never be reached.

<u>Proof</u>: According to the Cayley-Hamilton theorem, if

$$a(z) = \det(zI - A) = z^n + a_1 z^{n-1} + \ldots + a_n$$

denotes the characteristic polynomial of A, then

$$a(A) = A^n + a_1 A^{n-1} + \ldots + a_n I = 0 \quad . \tag{6}$$

This implies

$$A^n B = -a_1 A^{n-1} B \;\ldots\; -a_n B \tag{7}$$

so that the columns of $R_n = [B \; AB \ldots A^{n-1}B]$ span the columns of A^nB and thus of $\mathcal{R}_{n+1}$.

<u>Definition</u>: a) The reachable space $\mathcal{R}$ of the system (1) is the set of states

that can be reached in a finite number of steps. b) A system is reachable if $\mathcal{R} = \mathbb{R}^n$.

From the properties 1)-3) of the reachability spaces $\mathcal{R}_k$, we obtain the following result.

<u>Theorem 1</u>: a) The reachable space $\mathcal{R} = \mathcal{R}_n$, the space spanned by the columns of $R_n = [B \; AB \; \dots \; A^{n-1}B]$. b) The system (1) is reachable if and only if R_n has rank n.

The reachability matrix

$$R_n = [B \;\; AB \;\; \dots \;\; A^{n-1}B]$$

has size $n \times nm$, i.e. it has more columns than rows. Consequently it has rank n if and only if its n rows are linearly independent.

<u>Example</u>: Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(k) \quad .$$

Then the reachability matrix

$$R_2 = [B \;\; AB] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

has only rank 1, so that the system is not reachable. This can be seen by noting that the subsystem

$$x_1(k+1) = 2\,x_1(k)$$

is unaffected by $u(k)$. The space $\mathcal{R}$ of reachable states is spanned by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which spans the column space of R_2. ∎

<u>Definition</u>: A subspace V is said to be A-invariant if for all $\underline{v} \in V$ we have $A\underline{v} \in V$. This is denoted as $AV \subseteq V$.

<u>Geometric characterization of $\mathcal{R}$</u>: Two important geometric features of the reachable space $\mathcal{R}$ are as follows.

1) $\mathcal{B} \subseteq \mathcal{R}$: the vector space $\mathcal{B}$ spanned by the columns of B is contained in $\mathcal{R}$. This is due to the fact that B is a submatrix of the reachability matrix

$$R_n = [B \;\; AB \;\; \ldots \;\; A^{n-1}B]$$

whose columns span $\mathcal{R}$.

2) $A\mathcal{R} \subseteq \mathcal{R}$: $\mathcal{R}$ is A-invariant. To see this, assume that the state $\underline{v}$ can be reached in s steps and that the inputs taking $\underline{x}(0) =$ to $\underline{x}(s) = \underline{v}$ are $\underline{u}(0) \ldots \underline{u}(s-1)$. Then $A\underline{v}$ can be reached in $s+1$ steps by applying the input sequence $\underline{u}(0) \ldots \underline{u}(s-1)$, $\underline{u}(s) = \underline{0}$, so that $\underline{x}(s+1) = A\underline{v}$. Thus if $\underline{v}$ is reachable in a finite number of steps, so is

$A\underline{v}$, so that $\mathcal{R}$ is A-invariant.

In fact, $\mathcal{R}$ can be characterized as the smallest A-invariant space containing $\mathcal{B}$. This is due to the fact that if the columns of B belong to some A-invariant vector space V, then AB, $A^2B \ldots$ $A^kB \ldots$ must also be in V, so that $\mathcal{R} \subseteq V$. Thus any A-invariant space containing $\mathcal{B}$ must also contain $\mathcal{R}$.

This geometric characterization may appear esoteric, but it plays in fact an important role in deriving the decomposition of a system into reachable and unreachable parts.

<u>Similarity transformation</u>: First, we consider the effect of an arbitrary change of coordinates

$$\underline{x}(k) = T\,\underline{\tilde{x}}(k) \qquad (8)$$

on the DT system

$$\underline{x}(k+1) = A\,\underline{x}(k) + B\,\underline{u}(k) \qquad (9a)$$

$$\underline{y}(k) = C\,\underline{x}(k) + D\,\underline{u}(k)\,. \qquad (9b)$$

Substituting (8) inside (9a)-(9b) gives

$$\underline{\tilde{x}}(k+1) = \underbrace{T^{-1}AT}_{\tilde{A}}\,\underline{\tilde{x}}(k) + \underbrace{T^{-1}B}_{\tilde{B}}\,\underline{u}(k) \qquad (10a)$$

$$\underline{y}(k) = \underbrace{CT}_{\tilde{C}}\,\underline{\tilde{x}}(k) + D\,\underline{u}(k)\,, \qquad (10b)$$

so that under the similarity transformation (8), the DT system (A, B, C, D) is transformed into $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\tilde{A} = T^{-1} A T \quad , \quad \tilde{B} = T^{-1} B \quad , \quad \tilde{C} = C T \quad , \quad \tilde{D} = D \quad . \qquad (11)$$

An important fact is that the similarity transformation (8) preserves the system transfer function, i.e.

$$\tilde{H}(z) = \tilde{C}(zI - \tilde{A})^{-1} \tilde{B} + \tilde{D} = C(zI - A)^{-1} B + D = H(z) \quad . \qquad (12)$$

This is just a consequence of the fact that the similarity transformation $x(k) = T\tilde{x}(k)$ changes the internal coordinates of the system, but leaves the inputs $\underline{u}(k)$ and outputs $\underline{y}(k)$ invariant. Then since

$$\underline{Y}(z) = H(z)\, \underline{U}(z) = \tilde{H}(z)\, \underline{U}(z)$$

we must have $H(z) = \tilde{H}(z)$. This can also be verified directly by noting that

$$zI - \tilde{A} = T^{-1}(zI - A)\, T \, ,$$

so that $(zI - \tilde{A})^{-1} = T^{-1}(zI - A)^{-1} T$, and

$$\tilde{C}(zI - \tilde{A})^{-1} \tilde{B} + D = C \underbrace{T\,T^{-1}}_{I} (zI - A)^{-1} \underbrace{T\,T^{-1}}_{I} B + D$$

$$= C(zI - A)^{-1} B + D \, .$$

<u>Decomposition into reachable/unreachable parts</u>. Assume that the system (A, B) is not reachable. Then the reachability matrix

$R_n = [B \ AB \ \dots \ A^{n-1}B]$ has rank $\ell < n$. Let $\{\underline{t}_1 \dots \underline{t}_\ell\}$ be a basis for the column space $\mathcal{R}$ of R_n. This basis can always be completed into a basis for the whole space $\mathbb{R}^n$ by adding some additional basis vectors $\{\underline{t}_{\ell+1} \dots \underline{t}_n\}$. Then

$$T = [\underline{t}_1 \dots \underline{t}_\ell \ \underline{t}_{\ell+1} \dots \underline{t}_n]$$

is an invertible matrix.

Since the columns of B are contained in the reachable space $\mathcal{R}$, they can be expressed as a linear combination of the vectors $\underline{t}_1 \dots \underline{t}_\ell$, so that

$$B = \underbrace{[\underline{t}_1 \dots \underline{t}_\ell \ \underline{t}_{\ell+1} \dots \underline{t}_n]}_{T} \begin{bmatrix} B_r \\ \hdashline 0 \end{bmatrix} \begin{matrix} \updownarrow \ell \\ \updownarrow n-\ell \end{matrix}$$

Since $\mathcal{R}$ is A-invariant, the vectors $\{A\underline{t}_1 \dots A\underline{t}_\ell\}$ must belong to $\mathcal{R}$ and are therefore expressible as linear combinations of $\underline{t}_1 \dots \underline{t}_\ell$, so that

$$A \underbrace{[\underline{t}_1 \dots \underline{t}_\ell \ \underline{t}_{\ell+1} \dots \underline{t}_n]}_{T} = [\underline{t}_1 \dots \underline{t}_\ell \ \underline{t}_{\ell+1} \dots \underline{t}_n] \underbrace{\begin{bmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix}}_{\ell \quad n-\ell} \begin{matrix} \updownarrow \ell \\ \updownarrow n-\ell \end{matrix}$$

for some matrices A_r, $A_{\bar{r}}$ and $A_{r\bar{r}}$.

Thus, with the matrix T selected as above, under the similarity transformation $\underline{x}(k) = T\tilde{\underline{x}}(k)$, the system (A, B, C, D) becomes

$$\tilde{A} = T^{-1} A T = \left[\begin{array}{c|c} A_r & A_{r\bar{r}} \\ \hline 0 & A_{\bar{r}} \end{array}\right] \qquad \tilde{B} = T^{-1} B = \left[\begin{array}{c} B_r \\ \hline 0 \end{array}\right]$$

$$\tilde{C} = C T = [\, \underbrace{C_r}_{\ell} \;\; \underbrace{C_{\bar{r}}}_{n-\ell} \,] \qquad \tilde{D} = D \,. \tag{13}$$

The transformed system (13) has several important properties.

1) (A_r, B_r) is a <u>reachable pair</u>. To see this, note that

$$\tilde{R}_n = [\, \tilde{B} \;\; \tilde{A}\tilde{B} \; \dots \; \tilde{A}^{n-1}\tilde{B} \,] = \begin{array}{c} \left[\begin{array}{c} B_r \;\; A_r B_r \; \dots \dots \; A_r^{n-1} B_r \\ \hline 0 \end{array}\right] \end{array} \begin{array}{l} \updownarrow \ell \\ \updownarrow n-\ell \end{array}$$

$$= T^{-1} [\, B \;\; AB \; \dots \; A^{n-1}B \,] = T^{-1} R_n \,. \tag{14}$$

Since the row space of R_n has dimension ℓ, and is unaffected by premultiplication by an invertible matrix T^{-1}, we see that the row space of $\tilde{R}_n$ is identical to that of R_n, and has dimension ℓ, so that the ℓ rows of $[B_r \; A_r B_r \dots A_r^{n-1} B_r]$ are linearly independent. The column space of $[B_r \; A_r B_r \dots A_r^{n-1} B_r]$ has therefore dimension ℓ. But A_r is $\ell \times \ell$, so that the column space of $[B_r \; A_r B_r \dots A_r^{\ell-1} B_r]$ is identical to that of $[B_r \; A_r B_r \dots A_r^{n-1} B_r]$ where $n > \ell$. This implies

$$\operatorname{rank} [\, B_r \;\; A_r B_r \; \dots \; A_r^{\ell-1} B_r \,] = \ell \,,$$

so that (A_r, B_r) is reachable.

2) If $\tilde{\underline{x}}(k) = \left[\begin{array}{c} \underline{x}_r(k) \\ \hline \underline{x}_{\bar{r}}(k) \end{array}\right] \begin{array}{l} \updownarrow \ell \\ \updownarrow n-\ell \end{array}$, the states $\underline{x}_{\bar{r}}(k)$ are <u>unreachable</u>, since

their dynamics

$$\underline{x}_{\bar{r}}(k+1) = A_{\bar{r}}\, \underline{x}_{\bar{r}}(k) \qquad (15)$$

are unaffected by the inputs $\underline{u}(k)$. Thus, if $\underline{x}_{\bar{r}}(0) = \underline{0}$, we have $\underline{x}_{\bar{r}}(k) \equiv 0$ for all k, so that these states can never attain nonzero values.

3) The transfer function

$$H(z) = C(zI-A)^{-1}B + D = C_r(zI-A_r)^{-1}B_r + D \qquad (16)$$

This indicates that the original systems (A, B, C, D) is a <u>nonminimal</u> description of the transfer function $H(z)$, since it requires n states, whereas the description (A_r, B_r, C_r, D) requires only $\ell < n$ states. Note that this last description may still be nonminimal if the pair (C_r, A_r) is not observable, as will be shown later.

To prove (16) we note that under the similarity transformation $\underline{x}(k) = T\,\tilde{\underline{x}}(k)$, we have

$$H(z) = \tilde{C}(zI-\tilde{A})^{-1}\tilde{B} + D$$

with

$$(zI-\tilde{A})^{-1} = \left[\begin{array}{c|c} (zI-A_r)^{-1} & (zI-A_r)^{-1}A_{r\bar{r}}(zI-A_{\bar{r}})^{-1} \\ \hline 0 & (zI-A_{\bar{r}})^{-1} \end{array}\right]$$

so that

$$(zI-\tilde{A})^{-1}\tilde{B} = \left[\begin{array}{c} (zI-A_r)^{-1}B_r \\ \hline 0 \end{array}\right]$$

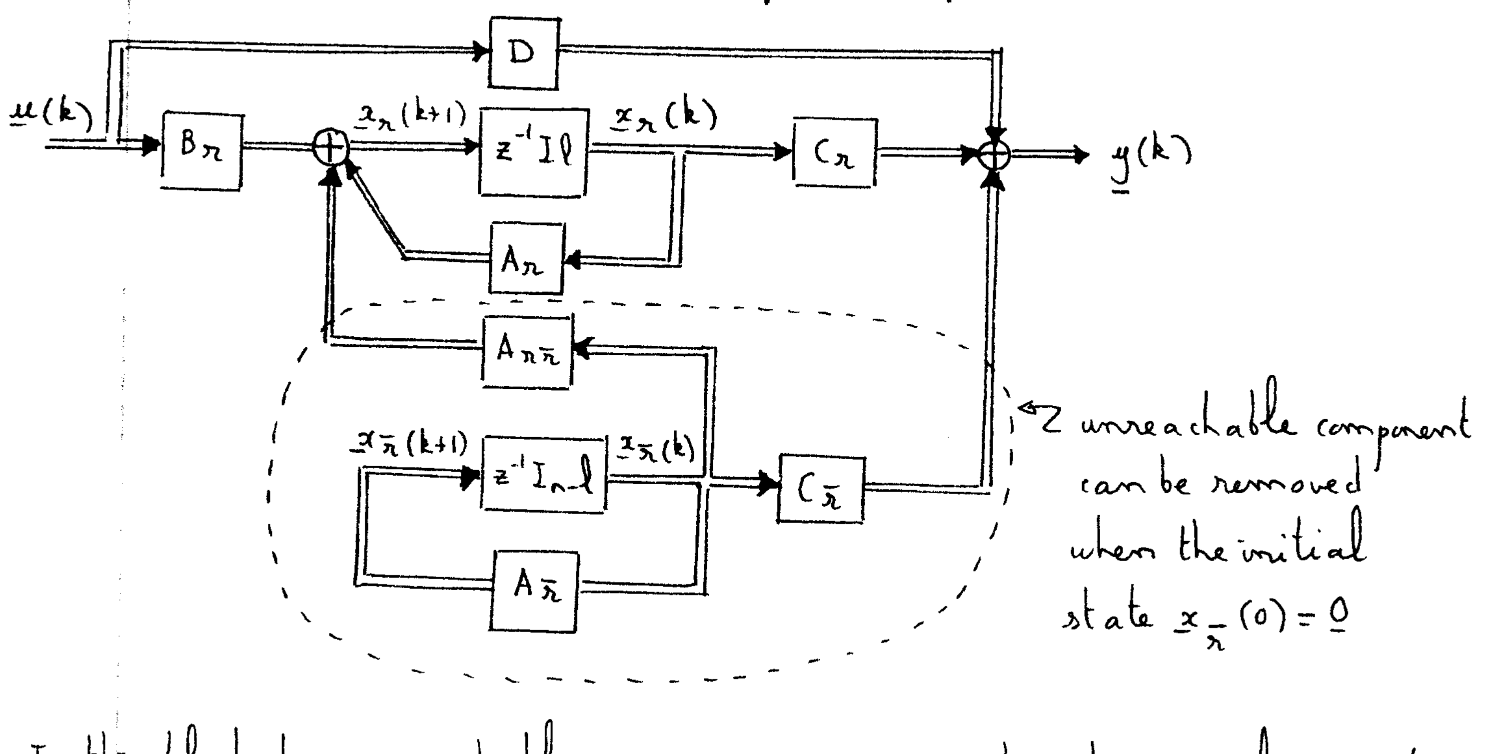
which upon premultiplication by $\tilde{C} = [\, C_r \;\; C_{\bar{r}} \,]$ yields (16).

<u>Block diagram representation</u>: In the new coordinates corresponding to $\underline{\tilde{x}}(k) = \begin{bmatrix} \underline{x}_r(k) \\ \underline{x}_{\bar{r}}(k) \end{bmatrix} = T^{-1}\underline{x}(k)$, the system dynamics

$$\begin{bmatrix} \underline{x}_r(k+1) \\ \underline{x}_{\bar{r}}(k+1) \end{bmatrix} = \begin{bmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix} \begin{bmatrix} \underline{x}_r(k) \\ \underline{x}_{\bar{r}}(k) \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} \underline{u}(k) \qquad (17a)$$

$$\underline{y}(k) = [\, C_r \;\; C_{\bar{r}} \,] \begin{bmatrix} \underline{x}_r(k) \\ \underline{x}_{\bar{r}}(k) \end{bmatrix} + D\,\underline{u}(k) \qquad (17b)$$

can be represented in block diagram form as follows.



In this block diagram, double arrows $\Longrightarrow$ represent vector signals, and block multipliers $\xrightarrow{\underline{v}} \boxed{M} \Longrightarrow \underline{w} = M\underline{v}$ represent matrix multiplications.

From this block diagram, we see that the unreachable state component $\underline{x}_{\bar{r}}(k)$ is not connected to the input $\underline{u}(k)$ and is not affected by $\underline{x}_r(k)$, so that it cannot be reached either directly from the input $\underline{u}(k)$ or through $\underline{x}_r(k)$. Given the initial condition $\underline{x}_{\bar{r}}(0) = \underline{0}$, the unreachable state component $\underline{x}_{\bar{r}}(k) \equiv 0$ for all k, so that it can be removed entirely from the block diagram. This is in particular the case for transfer function computations, since when evaluating the transfer function of a system we set all initial conditions equal to zero.

<u>Example</u>: Consider the system

$$\underline{x}(k+1) = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \underline{x}(k) + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underline{u}(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{C} \underline{x}(k).$$

This system has 2 inputs, 3 states and 1 output. Its reachability matrix

$$R_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

admits the column echelon form

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that rank $R_3 = 2 < 3$, i.e. the system is not reachable. The reachable space $\mathcal{R}$ is spanned by the columns of E, i.e.

$$\mathcal{R} = \text{space spanned by } \left\{ \underline{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \underline{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Consider now the similarity transformation T obtained by complementing the basis $\{\underline{t}_1, \underline{t}_2\}$ of $\mathcal{R}$ with a third vector, say $\underline{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, such that $\{\underline{t}_1, \underline{t}_2, \underline{t}_3\}$ is a basis of $\mathbb{R}^3$. We have

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

so that

$$\tilde{A} = T^{-1} A T = \left[\begin{array}{cc:c} \multicolumn{2}{c}{\overbrace{\quad\quad}^{A_r}} & \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & \underbrace{1}_{A_{\bar{r}}} \end{array}\right] \qquad \tilde{B} = T^{-1} B = \left.\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ \hdashline 0 & 0 \end{array}\right]\right\} B_r$$

$$\tilde{C} = C T = [\underbrace{2 \quad 1}_{C_r} \;\vdots\; \underbrace{1}_{C_{\bar{r}}}] \qquad \blacksquare$$

<u>Eigenvalue/eigenvector reachability test</u>: One problem with testing the reachability of a system by examining the rank of the matrix

$$R = [B \;\; AB \;\; \ldots \;\; A^{n-1}B]$$

is that the rank of a matrix is very sensitive to numerical errors. For example, if we consider

$$A = \begin{bmatrix} 1/2 & -1/4 \\ 0 & 1/4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the rank of

$$R = [\, b \;\; A b \,] = \begin{bmatrix} 1 & 1/4 \\ 1 & 1/4 \end{bmatrix}$$

is one, so that the system is unreachable. On the other hand if we consider the perturbed system (A_ε, b) with

$$A_\varepsilon = \begin{bmatrix} 1/2 & -1/4 \\ \varepsilon & 1/4 \end{bmatrix}$$

we have $R_\varepsilon = [\, b \;\; A_\varepsilon b \,] = \begin{bmatrix} 1 & 1/4 \\ 1 & 1/4+\varepsilon \end{bmatrix}$ which has rank 2. Thus a small perturbation of A or b may change dramatically the rank of A. This suggests that instead of deciding whether a system is reachable or not, we should try to measure "how reachable" it is. Specifically although (A_ε, b) is reachable certain states are very hard to reach, in the sense that the energy of the input signals required to reach these states is very large. This will lead us to introduce more quantitative reachability tests. A test which is not truly quantitative, but which provides some insight on lack of reachability of a system is the eigenvalue/eigenvector test, also known as PBH test after its inventors Popov, Belevitch and Hautus. This test can be stated as follows.

<u>Theorem 2</u>: A system (A, B) is reachable if and only if

$$\text{rank}\,[zI - A \,\vdots\, B] = n \tag{18}$$

for all complex values of z.

A useful observation that makes the above test practical is that the matrix $zI - A$, which is a submatrix of $[zI - A \,\vdots\, B]$, has full rank as long as z is not an eigenvalue of A, i.e. a zero of the characteristic polynomial $a(z) = \det(zI - A)$. This means that provided z is not an eigenvalue of A, $zI - A$ has n independent columns so that $[zI - A \,\vdots\, B]$ has rank n. The above test can therefore be replaced by the following <u>modified version</u>: a system (A, B) is reachable if and only if

$$\text{rank}\,[\lambda I - A \,\vdots\, B] = n \tag{19}$$

for all eigenvalues λ of A.

This test can be simplified further by examining what happens when (19) does not hold. When rank $[\lambda I - A \,\vdots\, B] < n$, the n rows of $[\lambda I - A \,\vdots\, B]$ must be linearly dependent, so that there exists a vector $\underline{q} \neq 0$ such that

$$\underline{q}^T [\lambda I - A \,\vdots\, B] = \underline{0} \,. \tag{20}$$

This means that

$$\underline{q}^T A = \lambda \underline{q}^T \qquad \underline{q}^T B = \underline{0} \ , \tag{21}$$

i.e. there exists a left eigenvector of A corresponding to the eigenvalue λ which is orthogonal to the columns of the matrix B. Thus the eigenvalue, eigenvector test reduces to: (A, B) is reachable if and only if the left eigenvectors $\underline{q}_i^T$ of A are such that

$$\underline{q}_i^T B \neq 0 \ .$$

In other words, the test requires computing all the left eigenvectors of A and checking whether they are orthogonal to all the columns of B or not.

To prove Theorem 2, we show

$$\text{rank}\,[zI - A \mid B] = n \text{ for all } z \Leftrightarrow \text{rank}\,[B \ AB \dots A^{n-1}B] = n \tag{22a}$$

or equivalently

$$\text{rank}\,[\lambda I - A \mid B] < n \text{ for some } \lambda \Leftrightarrow \text{rank}\,[B \ AB \dots A^{n-1}B] < n \ . \tag{22b}$$

We first prove the implication $\Rightarrow$. Assume that rank $[\lambda I - A \mid B] < n$ for some λ. Then there exists a left eigenvector $\underline{q}^T \neq 0$ of A such that $\underline{q}^T A = \lambda \underline{q}^T$ and $\underline{q}^T B = \underline{0}$. In this case $\underline{q}^T A^k = \lambda^k \underline{q}^T$ so that $\underline{q}^T A^k B = \lambda^k \underline{q}^T B = 0$ for all integers k. The vector $\underline{q}^T \neq 0$ satisfies therefore

$$\underline{q}^T [B \ AB \dots A^{n-1}B] = \underline{0} \ , \tag{23}$$

so that the rows of $R = [B\ AB \dots A^{n-1}B]$ are linearly dependent.

Next, we prove the implication $\Leftarrow$ in (22b). If the rank of the reachability matrix $R = [B\ AB \dots A^{n-1}B]$ is less than n, the system (A, B) is not reachable. By performing a similarity transformation, we can decompose it into its reachable and unreachable components, so that without loss of generality we can assume that

$$A = \begin{bmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix} \qquad B = \begin{bmatrix} B_r \\ 0 \end{bmatrix} .$$

Let now $\underline{q}_{\bar{r}}^T \neq 0$ be an arbitrary left eigenvector of the unreachable dynamics $A_{\bar{r}}$, i.e.

$$\underline{q}_{\bar{r}}^T A_{\bar{r}} = \lambda\, \underline{q}_{\bar{r}}^T . \tag{24a}$$

Then if

$$\underline{q}^T = [0 \quad \underline{q}_{\bar{r}}^T] \tag{24b}$$

we have

$$\underline{q}^T A = [0 \quad \underline{q}_{\bar{r}}^T] \begin{bmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{bmatrix} = [0 \quad \lambda \underline{q}_{\bar{r}}^T] = \lambda\, \underline{q}^T \tag{25a}$$

$$\underline{q}^T B = [0 \quad \underline{q}_{\bar{r}}^T] \begin{bmatrix} B_r \\ 0 \end{bmatrix} = 0 , \tag{25b}$$

so that $\underline{q}^T [\lambda I - A \,\vdots\, B] = 0$. This shows that if (A, B) is not reachable rank $[\lambda I - A \,\vdots\, B] < n$ for all the eigenvalues λ of the unreachable component $A_{\bar{r}}$ of the system. This proves Theorem 2.

Insert *

Example: For the system

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

that we considered earlier, we have

$$zI - A = \begin{bmatrix} z-1 & -1 & 0 \\ 0 & z-1 & 0 \\ 0 & -1 & z-1 \end{bmatrix} \qquad a(z) = \det(zI - A) = (z-1)^3 .$$

The left eigenvectors of A corresponding to $\lambda = 1$ are given by

$$[y_1 \ y_2 \ y_3] \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\lambda I - A} = [0 \ 0 \ 0] \Rightarrow y_1 + y_3 = 0 ,$$

so that A has two independent left eigenvectors $\underline{q}_1^T = [1 \ 0 \ -1]$ and $\underline{q}_2^T = [0 \ 1 \ 0$ associated to the eigenvalue $\lambda = 1$, which has multiplicity three. This implies that A has also a generalized left eigenvector corresponding to $\lambda = 1$. However for the PBH test only the left eigenvectors (not the generalized ones) need to be considered. We have

$$\underline{q}_1^T B = [0 \ 0] \qquad \underline{q}_2^T B = [1 \ 0] .$$

Since the left eigenvector $\underline{q}_1^T$ is orthogonal to the columns of B, the

<u>Remark</u>: The eigenvalue / eigenvector reachability test admits the following physical interpretation. Suppose there exists a left eigenvector $\underline{q} \neq$ of A which is orthogonal to the columns of B, i.e. such that

$$\underline{q}^T A = \lambda \underline{q}^T \qquad \underline{q}^T B = 0 .$$

Then, by

$$\begin{aligned} \underline{q}^T \underline{x}(k+1) &= \underline{q}^T A \underline{x}(k) + \underline{q}^T B \underline{u}(k) \\ &= \lambda \underline{q}^T \underline{x}(k) \end{aligned} \qquad (26)$$

This means that there is a linear combination $\underline{q}^T \underline{x}(k)$ of the states whose dynamics are unaffected by the inputs, so that if $\underline{q}^T \underline{x}(0) =$ we have $\underline{q}^T \underline{x}(k) = 0$ for all k, i.e. the component $\underline{q}^T \underline{x}(k)$ of the state vector is unreachable. In other words, any left eigenvector $\underline{q}$ of A which is perpendicular to the columns of B specifies an unreachable linear combination $\underline{q}^T \underline{x}(k)$ of the states.

system is not reachable.

<u>Reachability of the controller and controllability canonical forms</u>. Since the states of the controller and controllability canonical forms are easy to access from the input, we would expect that these realizations are reachable. This can be proved as follows.

<u>Controller form</u>: We have

$$zI - A_c = \begin{bmatrix} z+a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -1 & z & & & \\ & \ddots & \ddots & & 0 \\ 0 & & & z & \\ & & & -1 & z \end{bmatrix} \qquad b_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

and

$$[b_c \;\; zI - A_c] = \left[\begin{array}{cccc:c} 1 & z+a_1 & a_2 \cdots a_{n-1} & & a_n \\ & -1 & z & & \\ & 0 & \ddots & \ddots & \\ & & & z & \\ & & & -1 & z \end{array}\right] . \qquad (27)$$

Since the matrix $[b_c \;\; zI - A_c]$ contains the upper triangular matrix

$$\begin{bmatrix} 1 & z+a_1 & \cdots & a_{n-1} \\ -1 & z & & \\ & \ddots & \ddots & \\ 0 & & & z \\ & & & -1 \end{bmatrix}$$

as a submatrix, whose determinant is $(-1)^n$, which is independent of z, $[b_c \;\; zI - A_c]$ has full rank for all z, so that (A_c, b_c) is reachable.

<u>Controllability form</u>: The reachability matrix of

$$A_{co} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 & -a_2 \\ 0 & & & 1 & -a_1 \end{bmatrix} \qquad b_{co} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is $R_{co} = [\, b_{co} \;\; A_{co} b_{co} \;\; \ldots \;\; A_{co}^{n-1} b_{co} \,] = I_n$, the $n \times n$ identity matrix. Thus (A_{co}, b_{co}) is reachable.

<u>Observer form</u>: It is also of interest to check whether the observer realization of $H(z) = b(z)/a(z)$ with

$$a(z) = z^n + a_1 z^{n-1} + \ldots + a_n \qquad b(z) = b_1 z^{n-1} + \ldots + b_n$$

is reachable or not. It turns out this is not always guaranteed. Given

$$A_o = \begin{bmatrix} -a_1 & 1 & & & \\ -a_2 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & 1 \\ -a_n & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad b_o = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

it is easy to check that if λ is an eigenvalue of A_o, i.e. if $a(\lambda) = \det(\lambda I - A_o) = 0$, then

$$\underline{q}^T = [\lambda^{n-1} \ldots \lambda \; 1] \tag{28}$$

is the only left eigenvector of A_0 corresponding to λ. In other words, to each eigenvalue λ we can associate a single left eigenvector $\underline{q}^T$ with the structure (27). The observer realization will be reachable as long as for all such eigenvectors we have

$$\underline{q}^T b = b_1 \lambda^{n-1} + \ldots + b_n = b(\lambda) \neq 0 \; . \tag{29}$$

But if $b(\lambda) = 0$, this means that $z = \lambda$ is a common root of $a(z)$ and $b(z)$, so that there is a cancellation between $b(z)$ and $a(z)$. This gives the following result.

<u>Lemma</u>: The observer realization (A_0, b_0, c_0) is reachable provided there exists no cancellation between the numerator and denominator polynomials $b(z)$ and $a(z)$ of $H(z)$.