Midterm #1 Solutions

Problem 1

a) We perform elementary row operations on $A$ to obtain its reduced row echelon form $E_r$. To calculate the transforming matrix, we operate on

$$\begin{bmatrix}
A & I_3
\end{bmatrix}$$

instead of $A$ only. This gives

$$\begin{bmatrix}
A & I_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -3 & 2 & 1 & 0 & 0 \\
1 & 2 & -1 & 3 & 0 & 1 & 0 \\
2 & 0 & 10 & -2 & 0 & 0 & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 2 & -1 & 3 & 0 & 1 & 0 \\
0 & 1 & -3 & 2 & 1 & 0 & 0 \\
0 & -4 & 12 & -8 & 0 & -2 & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 2 & -1 & 3 & 0 & 1 & 0 \\
0 & 1 & -3 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 4 & -2 & 1
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 0 & 5 & -1 & -2 & 1 & 0 \\
0 & 1 & -3 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 4 & -2 & 1
\end{bmatrix} = [E_r | M] ,$$

where we have $M A = E_r$.

Row space: It is spanned by the nonzero columns of $E_r$, so that

$$R(A^T) = \text{space spanned by } \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix} , \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} .$$

Column space: The pivots are located in the first and second column of $E_r$, and the corresponding columns of $A$ span $R(A)$:

$$R(A) = \text{space spanned by } \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} , \begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix} .$$
Right null space: It is generated by expressing the nonpivot columns of $E_r$ as a linear combination of the pivot columns. This gives

$$\mathcal{N}(A) = \text{space spanned by } \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

Left null space: It is given by the rows of $M$ corresponding to the zero rows of $E_r$, so that

$$\mathcal{N}(A^T) = \text{space spanned by } \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}.$$ 

Clearly

$$\dim \mathcal{R}(A) = \dim \mathcal{R}(A^T) = 2.$$ 

$\dim \mathcal{N}(A) = 2$, and $\dim \mathcal{N}(A^T) = 1$.

b) For

$$\mathbf{y}_1 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix},$$

we have

$$M\mathbf{y}_1 = \mathbf{z}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \quad M\mathbf{y}_2 = \mathbf{z}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$ 

$A\mathbf{x} = \mathbf{y}_1$ has a two-dimensional space of solutions obtained by solving $E_r\mathbf{x} = \mathbf{z}_1$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix},$$

where $x_3$ and $x_4$ are arbitrary. On the other hand, since the third entry of $\mathbf{z}_2$ is nonzero, $A\mathbf{x} = \mathbf{y}_2$ has no solution.

**Problem 2:**

a) We have

$$sI_3 - A = \begin{bmatrix} s - 1 & 1 & 0 \\ 1 & s - 2 & 1 \\ 0 & 1 & s - 1 \end{bmatrix},$$

so that

$$a(s) = \det(sI_3 - A) = (s - 1)^2(s - 2) - 2(s - 1)$$

$$= (s - 1)[(s - 1)(s - 2) - 2] = s(s - 1)(s - 3).$$
A has therefore three distinct eigenvalues: \( \lambda_1 = 0, \lambda_2 = 1 \) and \( \lambda_3 = 3 \). The corresponding eigenvectors are evaluated as follows. For \( \lambda_1 = 0 \), the equation

\[
(\lambda_1 I_3 - A)x_1 = \begin{bmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

yields

\[ x_1 = x_2 = x_3 \]

so that

\[ x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T . \]

For \( \lambda_2 = 1 \), we have

\[
(\lambda_2 I_3 - A)x_2 = \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

which gives

\[ x_2 = 0 , \ x_3 = -x_1 , \]

so that

\[ x_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T . \]

Finally, for \( \lambda_3 = 3 \), we find

\[
(\lambda_3 I_3 - A)x_3 = \begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

so that

\[ x_2 = -2x_1 = -2x_3 \]

and thus

\[ x_3 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T . \]

The columns of the diagonalizing transformation \( T \) are the eigenvectors of \( A \), so

\[ T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
1 & 0 & -2 \\
1 & -1 & 1
\end{bmatrix}, \]

and

\[ \Lambda = \text{diag} \{ 0, 1, 3 \} . \]

b) The matrix \( B \) is in companion form, so that its characteristic polynomial is obtained by inspection, just by reading off the coefficients of the last row of \( B \):

\[ b(s) = \det (s I_3 - B) = s^3 + 4s^2 + 5s + 2 . \]
The polynomial $b(s)$ is zero for $s = -1$ and $s = -2$ and
\[ b(s) = (s + 1)^2(s + 2), \]
so $B$ has the eigenvalue $\lambda_1 = -1$ with multiplicity two, and $\lambda_2 = -2$ with multiplicity one. For $\lambda = -1$, we have
\[
(\lambda_1 I_3 - B)x_1 = \begin{bmatrix}
-1 & -1 & 0 \\
0 & -1 & -1 \\
2 & 5 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]
which yields
\[ x_1 = x_3 = -x_2 . \]
Thus $B$ has a single eigenvector
\[ x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}^T \]
corresponding to eigenvalue $\lambda_1 = -1$ of multiplicity two. This means it must have a generalized eigenvector $g$ specified by
\[
(B - \lambda_1 I_3)g = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
-2 & -5 & -3
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
g_3
\end{bmatrix} = \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix} = x_1 .
\]
We find
\[ g_1 + g_2 = 1 , \quad g_2 + g_3 = -1 , \]
so that
\[ g = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^T . \]
For $\lambda = -2$, we have
\[
(\lambda_2 I - B)x_2 = \begin{bmatrix}
-2 & -1 & 0 \\
0 & -2 & -1 \\
2 & 5 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]
which gives
\[ x_2 = -2x_1 , \quad x_3 = -2x_1 , \]
so that
\[ x_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}^T \]
is the eigenvector corresponding to $\lambda_2 = -2$.

d) Since $B$ has only one eigenvector $x_1$ corresponding to $\lambda_1 = -1$, even though $\lambda_1$ has multiplicity two, it must have a Jordan block of size two corresponding to $\lambda_1 = -1$. This implies that
\[
J = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

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is the Jordan form of $B$.

Problem 3:

a) By labeling the states from right to left as indicated in the block diagram, we obtain the state-space equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{align*}
\]

where

\[
x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T,
\]

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3/2 \\ -3 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\]

which is a realization in observability canonical form.

b) We have

\[
\begin{align*}
x_1(t) &= y(t) \quad x_2(t) = \dot{x}_1(t) = \dot{y}(t) \\
x_3(t) &= \ddot{x}_2(t) - \frac{3}{2}u(t) = \ddot{y}(t) - \frac{3}{2}y(t).
\end{align*}
\]

Substituting these relations inside

\[
\dot{x}_3 = -(x_1 + 2x_2 + 2x_3 + 3u)
\]

gives

\[
y^{(3)}(t) - \frac{3}{2}y^{(1)}(t) + 2y^{(2)}(t) - \frac{3}{2}u(t) + 2y^{(1)}(t) + y(t) = -3u(t)
\]

so that after simplifications we find that the input $u(t)$ and output $y(t)$ satisfy the third-order differential equation

\[
y^{(3)}(t) + 2y^{(2)}(t) + 2y^{(1)}(t) + y(t) = \frac{3}{2}y^{(1)}(t). \tag{1}
\]

The system transfer function is given by

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{3s/2}{s^3 + 2s^2 + 2s + 1} = \frac{3s/2}{(s + 1)(s^2 + s + 1)}.
\]

c) The block diagram (i) is just the controller canonical realization of (1). The block diagram (ii) is obtained by placing two controller realizations in parallel. Its transfer function is

\[
H_2(s) = \frac{3}{2} \left[ \frac{s + 1}{s^2 + s + 1} - \frac{1}{s + 1} \right] = \frac{3s/2}{(s + 1)(s^2 + s + 1)} = H(s).
\]
Thus it is equivalent to the block diagram of part a).

**Problem 4**

a) Consider the signal flow graph of Fig. 1.

![Signal Flow Graph](image)

**Figure 1: Signal flow graph.**

The variables corresponding to nodes $P_i$ with $1 \leq i \leq 3$ are given by

$P_1 = U(z) + \frac{1}{2} X_2 \quad P_2 = -Y(z)$

and

$P_3 = Y(z) = P_1 + \frac{3}{2} X_3 = \frac{1}{2} X_2 + \frac{3}{2} X_3 + U(z),$ 

so that

$x_1(k + 1) = p_1(k) = \frac{1}{2} x_2(k) + u(k)$

$x_2(k + 1) = -2 x_1(k) + p_2(k) = -2 x_1(k) - \frac{1}{2} x_2(k) - \frac{3}{2} x_3(k) - u(k)$

$x_3(k + 1) = p_2(k) = \frac{1}{2} x_2(k) - \frac{3}{2} x_3(k) - u(k).$

These equations can be written compactly in state-space form as

$x(k + 1) = A x(k) + B u(k)$

$y(k) = C x(k) + D u(k),$

with

$x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$

and

$A = \begin{bmatrix} 0 & 1/2 & 0 \\ -2 & -1/2 & -3/2 \\ 0 & -1/2 & -3/2 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 1/2 & 3/2 \end{bmatrix}$

$D = 1.$
b) The signal flow graph has three elementary loops shown in Fig. 2 below.

The loop gains for the three loops are

\[ L_1 = -z^{-2}, \quad L_2 = -\frac{3}{2}z^{-1} \quad \text{and} \quad L_3 = -\frac{1}{2}z^{-1}. \]

The loops \( L_1 \) and \( L_2 \) do not touch, so the determinant of the signal flow graph is

\[
D = 1 - (L_1 + L_2 + L_3) + L_1 L_2
= 1 + z^{-2} + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-1} + \frac{3}{2}z^{-3}
= 1 + 2z^{-1} + z^{-2} + \frac{3}{2}z^{-3}.
\]

There is only one direct path from the input to the output which is shown in Fig. 3.

![Diagram of signal flow graph](image)

\[ \text{Figure 3: Direct path between input node } U(z) \text{ and output node } Y(z). \]

Its gain is \( G_1 = 1 \). The path touches all three elementary loops at either \( P_1 \) or \( P_3 \), so its cofactor \( D_1 = 1 \) and according to Mason’s rule

\[
H(z) = \frac{G_1 D_1}{D} = \frac{1}{1 + 2z^{-1} + z^{-2} + \frac{3}{2}z^{-3}}.
\]