Problem Set #3 – due Wednesday, October 21, 2009

1. Write the state equations in terms of capacitor voltages and inductor currents for the circuit shown below. To do so, use the procedure described in class which consists in replacing capacitors and inductors by voltage sources and current sources, respectively, and then solving for the currents through the capacitors and the voltages across the inductors. Note that this procedure remains valid even if the circuit contains dependent sources. For circuits containing op-amps, assume that the op-amps are ideal.

2. Construct the controller and observer realizations of the differential equation

\[ \ddot{y} + 5\dot{y} + 6y = \ddot{u} + 2\dot{u} + u. \]

What is the characteristic polynomial of the corresponding A matrices?

3. This problem is concerned with three representations for a continuous-time system: (i) differential equations, (ii) block diagrams, and (iii) state-space equations.

a) Write the state-equations for the system shown below using the state variables \(x_1, x_2, x_3, x_4,\) and \(x_5\) appearing in this figure. That is, find matrices \(A, B, C\) and \(D\) such that

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

where

\[ x^T = [x_1 \ x_2 \ x_3 \ x_4 \ x_5]. \]
b) Write the state equations for the system of part a) without using $x_4$, that is use $x_1$, $x_2$, $x_3$, and $x_5$ only. Assume zero initial conditions for all five integrators.

c) Write the state equation for the differential equation shown below using the state variable $x_1 = z$, $x_2 = dz/dt$, and $x_3 = d^2z/dt^2$. The input of the system is $u(t)$ and its output is $z(t)$:

$$\frac{d^3}{dt^3}z(t) + 8 \frac{d^2}{dt^2}z(t) - 6 \frac{d}{dt}z(t) + 4z(t) = 3u(t).$$

d) For the following state equations, write down the differential equation. Note what happened in part c):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 5 & 9 & 7 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t).$$

4.

a) Solve the state-space equations

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t), \quad x(0) = x_0$$
\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \quad t \geq 0.
\]

These are the equations of a harmonic oscillator with unit mass and no damping.

b) Consider now the system described by the state equations

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
-2 \\
1
\end{bmatrix} u(t), \quad \begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} = \begin{bmatrix}
x_{10} \\
x_{20}
\end{bmatrix}
\]

\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

(i) Calculate \( x_1(t), x_2(t) \) and \( y(t) \) for \( t \geq 0 \).

(ii) What is the transfer function described by these state equations?

(iii) Does the transfer function give an adequate description of this system? Give reasons for your answer.

**MATLAB Exercise:** The MATLAB commands appearing below are contained in the Control Systems Toolbox.

a) Verify your answer to Problem 2 by using the MATLAB command \([A,B,C,D] = \text{tf2ss}(\text{num,den})\). Given a transfer function

\[
H(s) = \frac{n(s)}{d(s)},
\]

where the coefficients of the numerator and denominator polynomials \( n(s) \) and \( d(s) \) are represented by the vectors \text{num} and \text{den}, respectively, this command returns a state-space realization in controller canonical form.

b) Verify your answers to part d) of Problem 3 and part b)-(ii) of Problem 4 by using the MATLAB command \([\text{num,den}] = \text{ss2tf}(A,B,C,D)\). Given a state-space model, this command returns the numerator and denominator of its transfer function.

c) Use the \text{connect} and \text{blkbuild} MATLAB commands to verify your results of part a) of Problem 3.

Please print the results of your MATLAB session for parts a)-c).