1. Consider the state equation

\[
x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k).
\]

a) Is \( \xi = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}^T \) reachable? If yes, what is the minimum number of steps required to take the state from the zero state to \( \xi \)? What inputs need to be applied?

b) Determine all states that are reachable.

c) Determine all states that are unobservable.

2. The LTI system

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

with

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

is obtained as the result of linearizing the non-linear equations of motion of an orbiting satellite about a steady-state solution. The state \( x_1(t) \) is the radial perturbation \( \Delta r(t) \) of the satellite with respect to its orbit, and the state \( x_3(t) \) is the angular perturbation \( \Delta \theta(t) \). The input \( u_1(t) \) is the radial thrust, and \( u_2(t) \) is the tangential thrust.

a) Find the eigenvalues of \( A \). Is the system stable?

b) Is the system reachable?

c) If the radial thruster fails, is the system completely reachable?
d) If the tangential thruster fails, is the system completely reachable?

e) If

\[ y(t) = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}, \]

compute the transfer function matrix of the system. Determine whether the system is observable.

3. Consider the circuit shown below. Find a state-space model of the circuit, and determine whether it is reachable and observable.

4. An inverted pendulum of mass \( m \) is hinged at point \( A \), as shown in the figure below. A gyro with spin angular momentum \( h \) is attached to the pendulum but is free to rotate about the pendulum axis, as shown in the figure. The angle of rotation around this axis is denoted by \( \phi \). A control torque \( Q \) can be applied to the gyro from the pendulum. The equations of motion are

\[ I \ddot{\theta} = mgl\theta - h\dot{\phi} \quad \text{and} \quad J\ddot{\phi} = h\dot{\theta} + Q, \]

where \( I \) is the moment of inertia of the pendulum plus gyro about \( A \), \( J \) is the moment of inertia of the gyro about axis \( AC \), and \( C \) is the mass center of the pendulum plus gyro.

a) Compute the transfer function from \( Q(t) \) to \( \phi(t) \), and from \( Q(t) \) to \( \theta(t) \).

b) Show that the system is controllable by \( Q \), observable with \( \phi \), and unobservable with \( \theta \).

c) Show that the system is always unstable.