**Problem Set #7** — due Wednesday, December 2, 2009

1. Consider the following circuit.

![Circuit Diagram](image)

(a) By using the capacitor voltages as state variables, obtain a state-space model for this circuit.

(b) Perform a four part Kalman decomposition of this model into parts that are reachable and observable, reachable and unobservable, unreachable and observable, unreachable and unobservable.

(c) Specify which physical variables in the circuit are unreachable or unobservable, and provide a physical interpretation for their lack of reachability or observability.

2. Consider the series composition of two linear time-invariant systems shown in Fig. refseries.

![Series Composition Diagram](image)

**Figure 1:** Linear systems in series.
a) Construct an example where Systems 1 and 2 are minimal state-space realizations of single-input single-output transfer functions $H_1(s)$ and $H_2(s)$, respectively, such that the series composition of the two systems, as shown above, is not a minimal realization of the overall transfer function $H_2(s)H_1(s)$.

b) Now let

$$H_1(s) = \begin{bmatrix} \frac{s+1}{s+2} \\ \frac{1}{s+5} \end{bmatrix}, \quad H_2(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+3} \end{bmatrix}.$$ 

(i) Find a fourth order realization of the overall system that is reachable, but not observable.

(ii) Find a fourth order realization that is observable, but not reachable.

(iii) Find a minimal realization of the overall system.

(iv) Find a fourth order realization that is neither reachable nor observable.

3. Consider the continuous-time LTI system

$$\begin{align*}
\dot{x}(t) &= Ax(t) \\
y(t) &= Cx(t),
\end{align*}$$

where $x$ and $y$ are $n$- and $p$-dimensional vectors, and the pair $(C, A)$ is observable. Suppose there exists a constant $\mu > 0$ and a positive definite matrix $P$ such that the Lyapunov equation

$$A^TP + PA + 2\mu P + C^TC = 0$$
is satisfied. Show that all eigenvalues of $A$ have real parts less than $-\mu$. The parameter $\mu$ represents the stability margin of $A$.

4. This problem concerns the effect of feedback on reachability and observability. Let

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t).
\end{align*}$$

Under state feedback, the input $u(t)$ is replaced by

$$u(t) = Kx(t) + v(t),$$

where $v(t)$ denotes the new input. Similarly, under output feedback, $u(t)$ is expressed as

$$u(t) = Gy(t) + v(t).$$

a) Prove that if the system is reachable, under state feedback the system will remain reachable.
b) Unlike reachability, observability is not necessarily preserved by state feedback. To verify this fact, check that for

\[ A = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \]

\[ K = \begin{bmatrix} 0 \\ -\frac{1}{b_1} \\ 0 \end{bmatrix}, \]

the pair \((C, A)\) is observable, but the system obtained after state feedback is not observable.

c) Show that reachability is preserved under output feedback.

d) Show that observability is preserved under output feedback.

5. Consider the system in Jordan form

\[ \dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \]

which has a repeated unstable eigenvalue equal to 1. Do you think it is possible to stabilize this system by using state feedback? If yes, find the gain vector \( K \) such that closed-loop system has eigenvalues \(-1, -1, -2, -2, -2\).

6. Let

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

Find a \(2 \times 4\) feedback matrix \(K\) such that the eigenvalues of the closed-loop matrix \(A - BK\) are \(-1, -2\) and \(-1 \pm j\). To solve this problem you should first transform the input space so that each new input acts on the system independently of the other, and then use one input to bring the system to companion form, while using the other to allocate the poles.