Quickest Change Detection in Multiple On-Off Processes: Switching With Memory

Jia Ye, Qing Zhao

Department of Electrical and Computer Engineering
University of California, Davis, CA 95616

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Quickest Change Detection

Change Point $T_0$  
Declare at $T_d$

Detection Delay

$t$

$X_1$ $X_2$ $\cdots$ $X_{T_0-1}$ $X_{T_0}$ $X_{T_0+1}$ $\cdots$ $X_{T_d}$ $\cdots$

i.i.d. $\sim f_0(x)$  
i.i.d. $\sim f_1(x)$

► Quickest Detection: $\min \mathbb{E}[(T_d - T_0)^+]$ subject to $\Pr[T_d < T_0] \leq \zeta$

Detection Delay  
Reliability Constraint

► Tradeoff: Detection delay vs. detection reliability.
Quickest Change Detection

Change Point $T_0$  
Declare at $T_d$

Detection Delay

$X_1 X_2 \ldots X_{T_0-1} X_{T_0} X_{T_0+1} \ldots X_{T_d} \ldots$

i.i.d. $\sim f_0(x)$  
i.i.d. $\sim f_1(x)$

**Quickest Detection:** $\min \ E[(T_d - T_0)^+]$ subject to $\Pr[T_d < T_0] \leq \zeta$

- Bayesian: Shiryayev’61, Borovkov’98, Tartakovsky&Veeravalli’05.
Application in Cognitive Radio

Measurements: In busy states: i.i.d with distribution $f_0(x)$;
In idle states: i.i.d with distribution $f_1(x)$.

Stopping Time: At time $t = T_d$, the user declares an opportunity.

Quickest Detection: $\min \mathbb{E}[T_d]$ subject to $\Pr[Z(T_d) = \text{busy}] \leq \zeta$
Quickest Detection in Multiple On-Off Processes

Two Fundamental Differences:

- Channel occupancy is an on-off process with multiple change points.
- There are multiple channels available.
Infinite Channel Case

- Quickest Detection of Idle Periods in Multiple On-Off Processes:
  - Continue, switch, or declare?

- Tradeoffs:
  - Whether to declare: delay vs. reliability.
  - Whether to switch: loss of data vs. avoiding bad realizations.
Outline

- Infinite channel case
  - A decision-theoretic formulation
  - The optimal detection rule: a threshold policy

- Finite channel case: switching with memory
  - Basic structure of the optimal policy
  - A low-complexity threshold policy

- Simulation examples

- Conclusion and work in progress
Infinite Channel Case

- Infinite number of independent homogeneous on-off processes.
- Busy period: geometrically distributed with mean \( m_B = \frac{1}{p_B} \).
- Idle period: geometrically distributed with mean \( m_I = \frac{1}{p_I} \).
- Fraction of idle time: \( \lambda_0 = \frac{m_I}{m_I + m_B} \).
Infinite Channel Case

\[
\begin{align*}
\min E\left[ \sum_{l=1}^{L-1} T_s(l) + T_d(L) \right] & \quad \text{s.t.} \quad \Pr\left[ Z_L\left( \sum_{l=1}^{L-1} T_s(l) + T_d(L) \right) = \text{busy} \right] \leq \zeta \\
\text{Detection Time} & \quad \text{Reliability Constraint}
\end{align*}
\]
A POMDP Formulation

- **State Space:** 0 (busy), 1 (idle), \( \Delta \) (absorbing state)
- **Action Space:** S (Switch), C (Continue), D (Declare)
- **State Transition:**

![Diagram of state transitions]

- **Cost:**
  - Switch or Continue: \( 1 \)
  - Declare during a busy period: \( \gamma \)
A POMDP Formulation

- **A Sufficient Statistic:** the information state (belief)
  \[
  \lambda_t = \Pr[Z_t = \text{idle}|X_1, X_2, \ldots, X_t] \\
  \lambda_0 = \frac{m_I}{m_I + m_B}
  \]

- **Update of the Information State**
  \[
  \lambda_t = \begin{cases} 
  T(\lambda_0|x) & a(t-1) = S, \ X_t = x \\
  T(\lambda_{t-1}|x) & a(t-1) = C, \ X_t = x 
  \end{cases}
  \]

- **\(T(\lambda|x)\): updated information state based on the new measurement \(x\).**
  \[
  T(\lambda|x) \triangleq \frac{(\lambda\bar{p}_I + \bar{\lambda}p_B)f_1(x)}{(\lambda\bar{p}_I + \lambda p_B)f_1(x) + (\lambda p_I + \lambda\bar{p}_B)f_0(x)}.
  \]
A POMDP Formulation

Channel switching and change detection policy $\pi$:

$$\lambda_t \in [0, 1] \implies a(t) \in \{S, C, D\}, \text{ for each time } t.$$

Quickest change detection:

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{\pi}\left[\sum_{t=0}^{\infty} R_{\pi}(\lambda_t) \mid \lambda_0 = \frac{mI}{m_B + m_I}\right],$$

Cost
Infinite Channel Case: Value Functions

► \( V(\lambda_t) \): the minimum expected total cost-to-go when the current belief is \( \lambda_t \).

\[
V(\lambda_t) = \min \{ \underbrace{V_C(\lambda_t)}_{\text{Continue}}, \underbrace{V_S(\lambda_t)}_{\text{Switch}}, \underbrace{V_D(\lambda_t)}_{\text{Declare}} \}. 
\]

► \( V_C(\lambda_t) \): the minimum expected total cost-to-go if continue at \( t \).

\[
V_C(\lambda_t) = 1 + \underbrace{\int_x P(x; \lambda_t) \underbrace{V(T(\lambda_t|x))}_{\text{Pr[ observe } x \text{ under } \lambda_t \text{ ]}} \, dx}_{\text{Pr[ observe } x \text{ under } \lambda_t \text{ ]}}.
\]

► \( V_S(\lambda_t) \): the minimum expected total cost-to-go if switch at \( t \).

\[
V_S(\lambda_t) = 1 + \underbrace{\int_x P(x; \lambda_0) \underbrace{V(T(\lambda_0|x))}_{\text{Pr[ observe } x \text{ under } \lambda_0 \text{ ]}} \, dx}_{\text{Pr[ observe } x \text{ under } \lambda_0 \text{ ]}} = V_C(\lambda_0)
\]

► \( V_D(\lambda_t) \): the minimum expected total cost-to-go if declare at \( t \).

\[
V_D(\lambda_t) = (1 - \lambda_t)\gamma.
\]
The Optimal Detection Rule: A Threshold Policy

\[
V(\lambda_t) = \min \{ V_S(\lambda_t), V_C(\lambda_t), V_D(\lambda_t) \}.
\]

\[
\begin{align*}
\pi^*_\infty(\lambda_t) = & \begin{cases} 
S, & \lambda_t \in [0, \eta_s) \\
C, & \lambda_t \in [\eta_s, \eta_d) \\
D, & \lambda_t \in [\eta_d, 1]
\end{cases} \\
\eta_s = \lambda_0 \\
\eta_d = 1 - \zeta
\end{align*}
\]
Objective:

$$\min \mathbb{E}[T_d] \quad \text{s.t.} \quad \Pr[Z(T_d) = \text{busy}] \leq \zeta$$

Resulting POMDP:

$$\begin{align*}
\lambda_t \quad \Rightarrow \quad \Lambda(t) &= [\lambda_1(t), \ldots, \lambda_N(t)] \\
\{C, S, D\} \quad \Rightarrow \quad \{C_1, \ldots, C_N, D_1, \ldots, D_N\}
\end{align*}$$
Basic Structure Of The Optimal Policy

- Always declare on channel $i = \text{arg max}_j \{\lambda_j\}$

- Declaring threshold $\eta$ is monotonically increasing with $\lambda$

- $a^*(\lambda_1, \lambda_2) = C_1 \iff a^*(\lambda_2, \lambda_1) = C_2$
A Low-Complexity Threshold Policy

A threshold policy: $\hat{\pi}_N$:

- Continue on the channel with $\max \lambda$

- Declare on the channel with $\max \lambda$ when $\max \lambda \geq \eta(\Lambda^{-i})$

$$\hat{\pi}_N \rightarrow \pi^*_\infty \quad \text{as} \quad N \rightarrow \infty$$
Simulation Examples

- $f_0(x)$, $f_1(x)$: Gaussian with zero mean and different variances.

- $SNR = 10dB$.

- $\eta = 1 - \zeta$. 
Simulation Example

increase both $m_B$ and $m_I$ while keeping $\lambda_0$ fixed
Simulation Example

![Graph showing the comparison between Single-Channel Strategy, Full-Sensing, and \( \hat{\pi} \) with respect to the average detection time and number of channels.](image)

- **Single-Channel Strategy**
- **Full-Sensing**
- \( \hat{\pi} \)

**Axes:**
- **Y-axis:** Average Detection Time
- **X-axis:** Number of Channels

The graph demonstrates the trade-off between the number of channels and the average detection time for different sensing strategies.
Conclusion and Work in Progress

\[ \lambda_1 = \lambda_2 \]

\[ (0, 0) \]

Work in Progress:

- The optimality of the low complexity threshold policy \( \hat{\pi}_N \).
- The asymptotic optimality of setting \( \eta_d = 1 - \zeta \).