A Class of Restless Bandit Problems: Indexability and Optimality of Whittle’s Index

Qing Zhao

Department of Electrical and Computer Engineering
University of California at Davis

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History
Clinical Trial (Thompson’33)
Web Search (Radlinski&etal'08)
Playing Golf with Multiple Balls *(Dumitriu&etal'03)*
Multi-Agent System (Whittle’88, LeNy&etal’08)
Cognitive Radio (Zhao & et al' 05)

- $N$ independent Gilbert-Elliot channels:

Channel 1

Channel $N$

Oppportunities

$p_{00}$ (busy)

$p_{01}$

$p_{10}$

$p_{11}$

(idle)
Information vs. Immediate Payoff

“Bandit problems embody in essential form a conflict evident in all human action: information versus immediate payoff.”

— P. Whittle (1989)
Information vs. Immediate Payoff

A Two-Armed Bandit:

- Two coins with unknown bias $\theta_1, \theta_2$.
- Head: reward $= 1$; Tail: reward $= 0$.
- Objective: maximize total reward over $n$ flips.

An Example (Berry & Fristedt ’85):

- $\theta_1 = \frac{1}{2}, \theta_2 = \begin{cases} 1, & \text{with probability } \frac{1}{4} \\ 0, & \text{with probability } \frac{3}{4} \end{cases}$

- To gain immediate payoff: flip Coin 1 indefinitely.
- To gain information: flip Coin 2 initially.
Non-Bayesian Formulation

- $(\theta_1, \theta_2)$ are treated as unknown deterministic parameters.
- $V_n^\pi(\theta_1, \theta_2)$: total reward of policy $\pi$.
- $n \max_{\theta_{\text{max}}} \{\theta_1, \theta_2\}$: total reward if $(\theta_1, \theta_2)$ were known.
- The cost of learning (regret):

  \[ C_n^\pi(\theta_1, \theta_2) \triangleq n\theta_{\text{max}} - V_n^\pi(\theta_1, \theta_2) = (\theta_{\text{max}} - \theta_{\text{min}})\mathbb{E}[\text{time spent on } \theta_{\text{min}}] \]

- Objective: $\min_\pi C_n^\pi(\theta_1, \theta_2)$. 
Classic Results under Non-Bayesian Formulation

- Lai & Robbins' 85:
  \[ C_n^*(\theta_1, \theta_2) \sim \frac{\theta_{\text{max}} - \theta_{\text{min}}}{I(\theta_{\text{min}}, \theta_{\text{max}})} \log n \quad \text{as} \quad n \to \infty \]
  \( KL \text{ distance} \)

- Anantharam & Varaiya & Walrand' 87:
  - extension from single play to multiple plays.
  - extension from i.i.d to Markovian reward processes.
Classic and Recent Results under Non-Bayesian Formulation

- **Lai & Robbins’85:**

  \[ C_n^*(\theta_1, \theta_2) \sim \frac{\theta_{\max} - \theta_{\min}}{I(\theta_{\min}, \theta_{\max})} \log n \quad \text{as} \quad n \to \infty \]

  - **KL distance**

- **Anantharam, Varaiya, & Walrand’87:**
  - extension from single play to multiple plays.
  - extension from i.i.d to Markovian reward processes.

- **Liu & Zhao’09:**
  - extension to *distributed* multiple players
    - *(distributed decision-making using only local observations)*
  - decentralized policy achieving *the same* \( \log n \) order of the regret.
  - fairness among players.
Bayesian Formulation

- $(\theta_1, \theta_2)$ are random variables with prior distributions $(f_{\theta_1}, f_{\theta_2})$.

- Policy $\pi$: choose an arm based on $(f_{\theta_1}, f_{\theta_2})$ and the observation history.

- $R_\pi(t)$: the reward obtained at time $t$.

- The total discounted reward over an infinite horizon:

$$V_\pi(f_{\theta_1}, f_{\theta_2}) \triangleq \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \beta^t R_\pi(t) | (f_{\theta_1}, f_{\theta_2}) \right], \quad (0 < \beta < 1)$$

- Objective: $\max_\pi V_\pi(f_{\theta_1}, f_{\theta_2})$. 
Bandit and MDP

Multi-Armed Bandit as A Class of MDP: (Bellman’56)

- $N$ independent arms with fully observable states $[Z_1(t), \cdots, Z_N(t)]$.
- One arm is activated at each time.
- Active arm changes state (known Markov process); offers reward $R_i(Z_i(t))$.
- Passive arms are frozen and generate no reward.
Bandit and MDP

Multi-Armed Bandit as A Class of MDP: *(Bellman’56)*

- \( N \) *independent* arms with *fully observable* states \([Z_1(t), \ldots, Z_N(t)]\).
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*Why is sampling processes with unknown distributions an MDP?*

- The state of each arm is the *posterior* distribution \( f_{\theta_i}(t) \) (*information state*).
- For an active arm, \( f_{\theta_i}(t+1) \) is updated from \( f_{\theta_i}(t) \) and the new observation.
- For a passive arm, \( f_{\theta_i}(t+1) = f_{\theta_i}(t) \).
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Solving Multi-Armed Bandit using Dynamic Programming:

- Exponential complexity with respect to $N$. 
The Slow Propagation of the Breaking News

“A colleague of high repute asked an equally well-known colleague:

— ‘What would you say if you were told that the multi-armed bandit problem had been solved?’

— ‘Sir, the multi-armed bandit problem is not of such a nature that it can be solved.’

— P. Whittle (1989)
Gittins’s Index

The Index Structure of the Optimal Policy:  (Gittins: 1960’s)

- Assign each state of each arm a priority index.
- Activate the arm with highest current index value.

Complexity:

- Arm are strongly decomposable (1 N-dim to N 1-dim problems).
- Linear complexity with N.
- Polynomial (cubic) with the state space size of an individual arm (Varaiya & Walrand & Buyukkoc ‘85, Katta & Sethuraman ’04).

Extensions:  (Varaiya & Walrand & Buyukkoc ’85)

- From Markovian to non-Markovian dynamics.
Restless Bandit

Restless Multi-Armed Bandit: (Whittle’88)

- Activate $K$ arms simultaneously.
- Passive arms also change state and offer reward.

Structure of the Optimal Policy:

- Not yet found.

Complexity:

- PSPACE-hard (Papadimitriou&Tsitsiklis’99).
Whittle's Index

Whittle’s Index: (Whittle’88)

- Provide a subsidy $m$ for passivity whenever the arm is made passive.
- Whittle’s index: the subsidy $m$ that makes active and passive actions equally attractive at the current state.
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**Performance:**

- Optimal under relaxed constraint on the average number of active arms.
- Asymptotically optimal under certain conditions *(Weber&Weiss’90)*.
- Near optimal performance observed from extensive numerical examples.
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**Difficulties:**

- Existence (indexability) not guaranteed and difficult to check.
- Numerical index computation infeasible for uncountable state space.
- Optimality in finite regime difficult to establish.
Main Results
Multichannel Dynamic Access

- Each channel is considered as an arm.
- State of arm $i$: posterior probability that channel $i$ is idle.
  \[
  \omega_i(t) = \Pr[\text{channel } i \text{ is idle in slot } t \mid \text{observations } O(1), \ldots, O(t - 1)]
  \]
- The expected immediate reward for activating arm $i$ is $\omega_i(t)$
Markovian State Transition

If channel $i$ is activated in slot $t$:

$$\omega_i(t+1) = \begin{cases} p_{11}, & \text{if } O_i(t) = 1 \\ p_{01}, & \text{if } O_i(t) = 0 \end{cases}$$

If channel $i$ is made passive in slot $t$:

$$\omega_i(t+1) = \omega_i(t)p_{11} + (1 - \omega_i(t))p_{01}.$$
Indexability

A Single-Armed Bandit with Subsidy:

- Being active at state $\omega$: reward $= \omega$.
- Being passive at any state: reward $= m$.

Indexability:

- $\mathcal{P}(m)$: the passive set under subsidy $m$.
- Indexability: $\mathcal{P}(m)$ increases monotonically as $m$ increases from $-\infty$ to $\infty$. 

Indexable

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{indexable}
\caption{Indexable set $\mathcal{P}(m)$ with subsidy $m$.}
\end{figure}

Not Indexable

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{not_indexable}
\caption{Not indexable set $\mathcal{P}(m)$ with subsidy $m$.}
\end{figure}
Proof for Indexability: Value Functions

A Single-Armed Bandit with Subsidy:

- Being active at state $\omega$: reward $= \omega$.
- Being passive at any state: reward $= m$.

Value Functions:

- $V_m(\omega)$: max total discounted reward starting from state $\omega$
  
  $$V_m(\omega) = \max\{V_m(\omega; \text{active}), \ V_m(\omega; \text{passive})\}$$

- $V_m(\omega; \text{active})$: max total discounted reward if active at $\omega$
  
  $$V_m(\omega; \text{active}) = \omega + \beta(\omega V_m(p_{11}) + (1 - \omega)V_m(p_{01})) \quad (\text{linear in } \omega)$$

- $V_m(\omega; \text{passive})$: max total discounted reward if passive at $\omega$
  
  $$V_m(\omega; \text{passive}) = m + \beta V_m(\underbrace{\omega p_{11} + (1 - \omega)p_{01}}_{\text{updated state}}) \quad (\text{convex in } \omega)$$
A threshold policy is optimal for the single-armed bandit with subsidy.

It thus suffices to prove the threshold $\omega^*(m)$ with the subsidy $m$. 
Proof for Indexability: Total Passive Time

- A sufficient condition for $\omega^*(m) \nearrow$ with $m$:

$$\frac{d(V_m(\omega; \text{passive}))}{d_m}|_{\omega=\omega^*(m)} \geq \frac{d(V_m(\omega; \text{active}))}{d_m}|_{\omega=\omega^*(m)} \quad (\diamond)$$

- $\diamond$ is shown based on
  - $\frac{d(V_m(\omega))}{d_m} = \mathbb{E}[\text{total passive time}].$
  - properties of the first crossing time:

Positive Correlation:

Negative Correlation:
Structure of Whittle’s Index Policy

The Semi-Universal Structure of Whittle’s Index Policy:

- No need to compute the index.
- No need to know \( \{p_{01}, p_{11}\} \) other than their order.

\[ p_{11} \geq p_{01} \text{ (positive correlation):} \]

\[ p_{11} < p_{01} \text{ (negative correlation):} \]
Structure of Whittle’s Index Policy: Positive Correlation

- Stay with good \( (G) \) channels and leave bad \( (B) \) ones to the end of the queue.

\[ K = 3 \]

\[ t = 1 \]
### Structure of Whittle's Index Policy: Negative Correlation

- Stay with bad ($B$) channels and leave good ($G$) ones to the end of the queue.
- *Reverse* the order of unobserved channels.

<table>
<thead>
<tr>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1($G$)</td>
<td>2</td>
</tr>
<tr>
<td>2($B$)</td>
<td>N</td>
</tr>
<tr>
<td>3($G$)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

$K = 3$
Robustness of Whittle's Index Policy

- Automatically tracks model variations:

\[ p_{11} = 0.6, p_{01} = 0.1 \ (T \leq 5); \ p_{11} = 0.9, p_{01} = 0.4 \ (T > 5) \]
Optimality of Whittle’s Index Policy

Optimality for positively correlated channels:
- holds for general $N$ and $K$.
- holds for both finite and infinite horizon (discounted/average reward).

Optimality for negatively correlated channels:
- holds for all $N$ with $K = N - 1$.
- holds for $N = 2, 3$. 
Performance of Whittle’s Index Policy w.r.t. $K$

Constant approximation factor $\eta = \frac{\text{Performance of Whittle’s index policy}}{\text{Optimal performance}}$

$p_{11} < p_{01}$ (Negative Correlation)

\[
\begin{align*}
\eta &= 1, & K &= N - 1, N \\
\eta &\geq \max\left\{\frac{1}{2}, \frac{K}{N}\right\}, & \text{otherwise}
\end{align*}
\]
Performance of Whittle’s Index Policy w.r.t. $N$

- $V(N)$: the average reward achieved by Whittle’s index policy ($K = 1$).
- For $p_{11} \geq p_{01}$, $V(N)$ converges to a constant $\frac{\omega_0}{1 - p_{11} + \omega_0}$ at geometric rate $p_{11} - p_{01}$.
- For $p_{11} < p_{01}$, $V(N)$ approaches a constant $\frac{p_{10}^{(2)}}{E - p_{01}G}$ at geometric rate $(p_{11} - p_{01})^2$.

![Graph showing upper and lower bounds of throughput limit with $p_{11} = 0.8$, $p_{01} = 0.1$]
Inhomogeneous Channels

Whittle’s Index in Closed Form:

- **Positive correlation** \((p_{11} \geq p_{01})**:

\[
I(\omega) = \begin{cases} 
\omega, & \omega \leq p_{01} \quad \text{or} \quad \omega \geq p_{11} \\
\frac{\omega}{1-p_{11}+\omega}, & \omega_o \leq \omega < p_{11} \\
\frac{(\omega-T^1(\omega))(L+2)+T^{L+1}(p_{01})}{1-p_{11}+(\omega-T^1(\omega))(L+1)+T^{L+1}(p_{01})}, & p_{01} < \omega < \omega_o
\end{cases}
\]

- **Negative correlation** \((p_{11} < p_{01})**:

\[
I(\omega) = \begin{cases} 
\omega, & \omega \leq p_{11} \quad \text{or} \quad \omega \geq p_{01} \\
\frac{p_{01}}{1+p_{01}-\omega}, & T^1(p_{11}) \leq \omega < p_{01} \\
\frac{p_{01}}{1+p_{01}-T^1(p_{11})}, & \omega_o \leq \omega < T^1(p_{11}) \\
\frac{\omega+p_{01}-T^1(\omega)}{1+p_{01}-T^1(p_{11})+T^1(\omega)-\omega}, & p_{11} < \omega < \omega_o
\end{cases}
\]
Solving for Whittle’s Index in Closed-Form

► Whittle’s index $I(\omega)$:

$$I(\omega) = \{ m : V_m(\omega; \text{active}) = V_m(\omega; \text{passive}) \}$$

► Need to obtain $V_m(\omega; \text{active})$ and $V_m(\omega; \text{passive})$ in closed-form

$$V_m(\omega; \text{active}) = \omega + \beta(\omega V_m(p_{11}) + (1 - \omega)V_m(p_{01}))$$

$$V_m(\omega; \text{passive}) = m + \beta V_m(\omega p_{11} + (1 - \omega)p_{01})$$

► $V_m(\omega; \text{active})$ and $V_m(\omega; \text{passive})$ can be written as functions of $\{V_m(p_{11}), V_m(p_{01})\}$ based on the threshold optimal policy and properties of the first crossing time.

► $\{V_m(p_{11}), V_m(p_{01})\}$ can be obtained in closed-form.
Properties of Whittle’s Index

Positive correlation ($p_{11} \geq p_{01}$):

Negative correlation ($p_{11} < p_{01}$):

Monotonicity $\Rightarrow$ equivalence with myopic policy $\Rightarrow$ structure and optimality.
Performance for Inhomogeneous Channels

- The tightness of the performance upper bound ($O(N(\log N)^2)$ running time).
- The near-optimal performance of Whittle's index policy

![Graph showing the relationship between K and the discounted total reward. The graph illustrates the performance of Whittle's index policy compared to the upper bound of the optimal policy.](image-url)
Conclusion and Acknowledgement
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- Indexability and Whittle’s index in closed-form: *K.Liu&Q.Zhao:08.*
- The semi-universal structure of Whittle’s index policy:
  - Equivalence to the myopic policy: *K.Liu&Q.Zhao:08.*
  - Structure of the myopic policy: *Q.Zhao&B.Krishnamachari:07.*
- The optimality of Whittle’s index policy:
  - Equivalence to the myopic policy: *K.Liu&Q.Zhao:08.*
  - Optimality of the myopic policy for $N = 2$: *Q.Zhao&B.Krishnamachari:07.*
  - Optimality of the myopic policy for $N > 2$:
- The performance of Whittle’s index policy for non-identical arms:
  - Scaling behavior w.r.t. $N$: *K.Liu&Q.Zhao:08.*
  - $O(N(\log N)^2)$-time algorithm for computing an upper bound: *K.Liu&Q.Zhao:08.*