When to Quit for A New Job: Quickest Detection of Spectrum Opportunities in Multiple Channels

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Quickest Detection of Spectrum Opportunities

Measurements: \( \{X_1, X_2, \ldots, X_{T_0-1}\} \) are i.i.d with distribution \( f_0(x) \);
\( \{X_{T_0}, X_{T_0+1}, \ldots\} \) are i.i.d with distribution \( f_1(x) \).

Stopping Time: At time \( t = T_d \), the user declares an opportunity.

Quickest Detection: \( \min \ E[(T_d - T_0)^+] \) subject to \( \Pr[T_d < T_0] \leq \zeta \)

Tradeoff: Detection delay vs. detection reliability.
Quickest Detection in Multiple Channels

- Quickest Detection:

  Continue in the current channel or seek for opportunities in a new channel?
Climbing The Corporate Ladder

Keep climbing?
Climbing The Corporate Ladder

Keep climbing?

Or look for a greener pasture?
Outline

► Quickest opportunity detection in a single channel
  □ Shiryayev’s algorithm for quickest detection

► Quickest opportunity detection in multiple channels
  □ A decision-theoretic formulation
  □ A threshold policy

► Simulation Examples

► Conclusion
**Quickest Detection In A Single Channel**

- **Measurements:** \( \{X_1, X_2, \ldots, X_{T_0-1}\} \) are i.i.d with distribution \( f_0(x) \);
  \( \{X_{T_0}, X_{T_0+1}, \ldots\} \) are i.i.d with distribution \( f_1(x) \).

- **Stopping Time:** At time \( t = T_d \), the user declares an opportunity.

- **Quickest Detection:**
  \[
  \min \mathbb{E}[(T_d - T_0)_{+}] \quad \text{subject to} \quad \mathbb{P}[T_d < T_0] \leq \zeta
  \]
  Detection Delay
  Interference Constraint

- **Tradeoff:** Detection delay vs. detection reliability.
Bayesian Formulation of Quickest Detection

Bayesian Formulation:

- Priori distribution of change point $T_0$: geometric

\[
\begin{align*}
\Pr[T_0 = 0] &= \lambda_0 \\
\Pr[T_0 = k] &= (1 - \lambda_0)p(1 - p)^{k-1}, \ \forall k > 0,
\end{align*}
\]
Shirayev’s Algorithm

A sufficient statistic: a posterior probability that change has already happened

\[ \lambda_t \triangleq P_r[T_0 \leq t \mid X_1, X_2, \ldots, X_t] \]

Shirayev’s detection rule:

\[ T_d = \inf \{ t : \lambda_t \geq \eta_d \} \]

Detection threshold \( \eta_d \): determined by the reliability constraint \( \zeta \).

Setting \( \eta_d = 1 - \zeta \) is asymptotically optimal as \( \zeta \to 0 \).
Quickest Detection In Multiple Channel

- A large number of homogeneous channels.
- Busy period: geometrically distributed with mean $m_B = \frac{1}{p_B}$.
- Idle period: geometrically distributed with mean $m_I = \frac{1}{p_I}$.
- Fraction of idle time: $\lambda_0 = \frac{m_I}{m_I + m_B}$. 
Quickest Detection In Multiple Channels

Channel 1

\[ T_S(1) \]

Switch

Channel 2

\[ T_S(2) \]

Switch

\[ \vdots \]

\[ T_S(L - 1) \]

Switch

Channel \( L \)

\[ T_D(L) \]

- Measurements: \( X \sim f_0(x) \) in busy periods; \( X \sim f_1(x) \) in idle periods.

\[
\min \mathbb{E} \left[ \sum_{l=1}^{L-1} T_s(l) + T_d(L) \right] \quad \text{s.t.} \quad \Pr \left[ \sum_{l=1}^{L-1} T_s(l) + T_d(L) = \text{busy} \right] \leq \zeta
\]

Waiting Time

Interference Constraint
A POMDP Formulation

- **State Transition:**

- **Cost:**
  - Switch or Continue: 1
  - Declare during a busy period: $\gamma$
A POMDP Formulation

- **A Sufficient Statistic:** the information state (belief)
  \[
  \lambda_t = \Pr[Z_t = \text{idle}|X_1, X_2, \ldots, X_t] \\
  \lambda_0 = \frac{m_I}{m_I + m_B}
  \]

- **Updates of the Information State**
  \[
  \lambda_t = \begin{cases} 
  T(\lambda_0|x) & a(t-1) = S, \ X_t = x \\
  T(\lambda_{t-1}|x) & a(t-1) = C, \ X_t = x
  \end{cases}
  \]

- **$T(\lambda|x)$:** updated information state based on the new measurement $x$.
  \[
  T(\lambda|x) \triangleq \frac{(\lambda \tilde{p}_I + \tilde{\lambda} p_B) f_1(x)}{(\lambda \tilde{p}_I + \tilde{\lambda} p_B) f_1(x) + (\lambda p_I + \tilde{\lambda} p_B) f_0(x)}.
  \]
A POMDP Formulation

- Channel switching and change detection policy $\pi$:

$$\lambda_t \in [0, 1] \implies a(t) \in \{S, C, D\}, \text{ for each time } t.$$

- Quickest change detection:

$$\pi^* = \arg \min_{\pi} \mathbb{E}_{\pi} \left[ \sum_{t=0}^{\infty} R_{\pi}(\lambda_t) \mid \lambda_0 = \frac{m_I}{m_B + m_I} \right].$$
Quickest Change Detection: Value Functions

- \( V(\lambda_t) \): the minimum expected total cost when the current info. state is \( \lambda_t \).
  \[
  V(\lambda_t) = \min\{ V_C(\lambda_t), V_S(\lambda_t), V_D(\lambda_t) \}. 
  \]

- \( V_C(\lambda_t) \): the minimum expected total cost if continue.
  \[
  V_C(\lambda_t) = 1 + \int_x P(x; \lambda_t) \ V(T(\lambda_t|x)) \ dx 
  \]

- \( V_S(\lambda_t) \): the minimum expected total cost if switch.
  \[
  V_S(\lambda_t) = 1 + \int_x P(x; \lambda_0) \ V(T(\lambda_0|x)) \ dx = V_C(\lambda_0) 
  \]

- \( V_D(\lambda_t) \): the minimum expected total cost if declare.
  \[
  V_D(\lambda_t) = (1 - \lambda_t) \gamma. 
  \]
Quickest Change Detection: A Threshold Policy

- $V_C(\lambda_t)$ is concave.

- $V_S(\lambda_t) = V_C(\lambda_0)$ \(\lambda_0 = \frac{m_I}{m_I+m_B}\)

- $V_D(\lambda_t)$ is linear.
Quickest Change Detection: A Threshold Policy

The switching threshold $\eta_s = \lambda_0 = \frac{m_I}{m_B + m_I}$, the fraction of the idle time;

The detection threshold $\eta_d = 1 - \zeta$ (asymptotically optimal as $\zeta \to 0$).
Simulation Examples

- \( f_0(x), f_1(x) \): Gaussian with zero mean and different variances.

- \( SNR = 10dB \).

- \( \eta_d = 1 - \zeta \).
Simulation Examples

A small number of channel switchings lead to significant reduction in detection time.
Simulation Examples

- Increase both $m_B$ and $m_I$ while keeping $\lambda_0$ fixed
Simulation Examples

- The expected time to catch an opportunity vs. the interference constraint $\zeta$:
Climbing The Corporate Ladder

Keep climbing?

Or look for a greener pasture?
Conclusions

Quickest Opportunity Detection in Multiple Channels

- A POMDP formulation.
- A simple threshold structure of the optimal detection rule.