Quickest Change Detection in Multiple On-Off Processes

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**Quickest Change Detection**

Change Point $T_0$ \hspace{2cm} Declare at $T_d$

Detection Delay

$t$

$X_1 \quad X_2 \quad \cdots \quad X_{T_0-1} \quad X_{T_0} \quad X_{T_0+1} \quad \cdots \quad X_{T_d} \quad \cdots$

i.i.d. $\sim f_0(x)$ \hspace{2cm} i.i.d. $\sim f_1(x)$

**Quickest Detection:** min $\mathbb{E}[(T_d - T_0)^+]$ subject to $\Pr[T_d < T_0] \leq \zeta$

**Tradeoff:** Detection delay vs. detection reliability.
Quickest Change Detection

Change Point $T_0$  
Declare at $T_d$

Detection Delay

$X_1 \quad X_2 \quad \cdots \quad X_{T_0-1} \quad X_{T_0} \quad X_{T_0+1} \quad \cdots \quad X_{T_d} \quad \cdots$

i.i.d. $\sim f_0(x)$  
i.i.d. $\sim f_1(x)$

Quickest Detection: $\min \mathbb{E}[\{T_d - T_0\}^+]$ subject to $\Pr[T_d < T_0] \leq \zeta$

- Bayesian: Shiryaev’61, Borovkov’98, Tartakovsky & Veeravalli’05.
Application in Cognitive Radio

- **Measurements:** \( \{X_1, X_2, \ldots, X_{T_0-1}\} \) are i.i.d with distribution \( f_0(x) \);
  \( \{X_{T_0}, X_{T_0+1}, \ldots\} \) are i.i.d with distribution \( f_1(x) \).

- **Stopping Time:** At time \( t = T_d \), the user declares an opportunity.

- **Quickest Detection:** \( \min \frac{\mathbb{E}(T_d - T_0)}{+} \) subject to \( \frac{\Pr[T_d < T_0]}{\leq \zeta} \)

Detection Delay

Interference Constraint
Quickest Detection in Multiple On-Off Processes

- Two Fundamental Differences:
  - Channel occupancy is an on-off process with multiple change points.
  - There are multiple channels available.
Quickest Detection of Idle Periods in Multiple On-Off Processes:

- Continue, switch, or declare?

Tradeoffs:

- Whether to declare: delay vs. reliability.
- Whether to switch: loss of data vs. avoiding bad realizations.
Outline

- Quickest change detection in a single stochastic process
  - Shiryaev’s algorithm

- Quickest detection in multiple on-off processes
  - A decision-theoretic formulation
  - The optimal detection rule: a threshold policy

- Simulation examples

- Conclusion and work in progress
Quickest Change Detection: Classic Bayesian Formulation

Bayesian Formulation:

- Prior distribution of change point $T_0$: geometric

\[
\Pr[T_0 = 0] = \lambda_0 \\
\Pr[T_0 = k] = (1 - \lambda_0)p(1 - p)^{k-1}, \forall k > 0,
\]
Shiryaev’s Algorithm

A sufficient statistic: \textbf{a posterior probability} that change has occurred
\[ \lambda_t \triangleq \Pr[T_0 \leq t | X_1, X_2, \ldots, X_t]. \]

Shiryaev’s detection rule:
\[ T_d = \inf\{t : \lambda_t \geq \eta_d\} \]

Detection threshold \( \eta_d \): determined by the reliability constraint \( \zeta \).

Setting \( \eta_d = 1 - \zeta \) is asymptotically optimal as \( \zeta \to 0 \).
A large number of independent homogeneous on-off processes.

- **Busy period**: geometrically distributed with mean $m_B = \frac{1}{p_B}$.
- **Idle period**: geometrically distributed with mean $m_I = \frac{1}{p_I}$.
- **Fraction of idle time**: $\lambda_0 = \frac{m_I}{m_I + m_B}$. 
Quickest Detection In Multiple On-Off Processes

\[ \min \mathbb{E}[\sum_{l=1}^{L-1} T_s(l) + T_d(L)] \quad s.t. \quad \Pr[Z_L(\sum_{l=1}^{L-1} T_s(l) + T_d(L)) = \text{busy}] \leq \zeta \]

Detection Time

Reliability Constraint
A POMDP Formulation

► **State Space:** 0 (busy), 1 (idle), \( \triangle \) (absorbing state)

► **Action Space:** S (Switch), C (Continue), D (Declare)

► **State Transition:**

- Transition diagram: [Diagram showing state transitions]

► **Cost:**

- Switch or Continue: 1
- Declare during a busy period: \( \gamma \)
A POMDP Formulation

- **A Sufficient Statistic**: the information state (belief)

  \[ \lambda_t = \Pr[Z_t = \text{idle}|X_1, X_2, \ldots, X_t] \]

  \[ \lambda_0 = \frac{m_I}{m_I + m_B} \]

- **Update of the Information State**

  \[ \lambda_t = \begin{cases} 
  T(\lambda_0 | x) & a(t-1) = S, \ X_t = x \\
  T(\lambda_{t-1} | x) & a(t-1) = C, \ X_t = x 
  \end{cases} \]

- **\( T(\lambda | x) \)**: updated information state based on the new measurement \( x \).

  \[ T(\lambda | x) \triangleq \frac{(\lambda \tilde{p}_I + \lambda \tilde{p}_B) f_1(x)}{(\lambda \tilde{p}_I + \lambda \tilde{p}_B) f_1(x) + (\lambda p_I + \lambda \tilde{p}_B) f_0(x)}. \]
A POMDP Formulation

- Channel switching and change detection policy $\pi$:

\[ \lambda_t \in [0, 1] \implies a(t) \in \{S, C, D\}, \text{ for each time } t. \]

- Quickest change detection:

\[ \pi^* = \arg \min_{\pi} \mathbb{E}_{\pi} \left[ \sum_{t=0}^{\infty} R_{\pi}(\lambda_t) \mid \lambda_0 = \frac{m_I}{m_B + m_I} \right], \]

where $\lambda_t$ represents the channel state at time $t$. The minimization is performed over all possible policies $\pi$, and the expected reward $R_{\pi}(\lambda_t)$ is calculated for each policy. The optimal policy $\pi^*$ is the one that minimizes the expected cumulative reward, given the initial state $\lambda_0$. This formulation is particularly useful in scenarios where the system needs to adapt quickly to changes in the environment while minimizing the cost associated with switching channels or detecting changes.
**Quickest Change Detection: Value Functions**

- **$V(\lambda_t)$**: the minimum expected total cost-to-go when the current belief is $\lambda_t$.
  
  $V(\lambda_t) = \min\{V_C(\lambda_t), V_S(\lambda_t), V_D(\lambda_t)\}$.  
  
  **Continue**  **Switch**  **Declare**

- **$V_C(\lambda_t)$**: the minimum expected total cost-to-go if continue at $t$.
  
  $V_C(\lambda_t) = 1 + \int_x P(x; \lambda_t) V(T(\lambda_t|x)) dx$

  Pr[ observe $x$ under $\lambda_t$ ]

- **$V_S(\lambda_t)$**: the minimum expected total cost-to-go if switch at $t$.
  
  $V_S(\lambda_t) = 1 + \int_x P(x; \lambda_0) V(T(\lambda_0|x)) dx = V_C(\lambda_0)$  

  Pr[ observe $x$ under $\lambda_0$ ]

- **$V_D(\lambda_t)$**: the minimum expected total cost-to-go if declare at $t$.
  
  $V_D(\lambda_t) = (1 - \lambda_t)\gamma$.  

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Quickest Change Detection: A Threshold Policy

- $V_D(\lambda_t)$ is linear.
- $V_C(\lambda_t)$ is monotonically decreasing (if $\frac{1}{m_B} + \frac{1}{m_I} \leq 1$) and concave.
- $V_S(\lambda_t) = V_C(\lambda_0)$, where $\lambda_0 = \frac{m_I}{m_I + m_B}$. 

![Graph showing $V_D(\lambda_t)$, $V_C(\lambda_t)$, and $V_S(\lambda_t)$ with threshold $\lambda_0$.]
Quickest Change Detection: A Threshold Policy

\[ V(\lambda_t) = \min \{ V_S(\lambda_t), V_C(\lambda_t), V_D(\lambda_t) \}. \]

Switch \quad Continue \quad Declare
Simulation Examples

- $f_0(x), f_1(x)$: Gaussian with zero mean and different variances.

- $SNR = 10dB$.

- $\eta_d = 1 - \zeta$. 

Simulation Example: Geometric Distribution

- Increase both $m_B$ and $m_I$ while keeping $\lambda_0$ fixed
Simulation Example: Arbitrary Distributions

- Busy period: Pareto distribution with increasing tail index
Conclusion and Work in Progress

Quickest Detection in Multiple On-Off Processes:

Work in Progress:

- Asymptotic optimality for arbitrary distributions and non-i.i.d. data.
- Minimax formulation.