Quickest Change Detection in Multiple On-Off Processes

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Quickest Change Detection

\[ \begin{align*}
\text{Change Point } T_0 & \quad \text{Declare at } T_d \\
\text{Detection Delay} & \\
& \\
X_1 & \quad X_2 & \quad \ldots & \quad X_{T_0-1} & \quad X_{T_0} & \quad X_{T_0+1} & \quad \ldots & \quad X_{T_d} & \quad \ldots \\
\text{i.i.d. } & \sim f_0(x) & \quad \text{i.i.d. } & \sim f_1(x) \\
& \\
\text{Quickest Detection: } \min & \quad \mathbb{E}[(T_d - T_0)^+] & \quad \text{subject to} & \quad \Pr[T_d < T_0] \leq \zeta \\
& \quad \text{Detection Delay} & \quad \text{Reliability Constraint} \\
\text{Tradeoff: } & \quad \text{Detection delay vs. detection reliability.}
\end{align*} \]
Quickest Change Detection

Change Point $T_0$  
Declare at $T_d$

Detection Delay

$X_1$  $X_2$  $\cdots$  $X_{T_0-1}$  $X_{T_0}$  $X_{T_0+1}$  $\cdots$  $X_{T_d}$  $\cdots$

$i.i.d. \sim f_0(x)$  
$i.i.d. \sim f_1(x)$

**Quickest Detection**: $\min \mathbb{E}[\left(T_d - T_0\right)^+]$ subject to $\Pr[T_d < T_0] \leq \zeta$

- **Bayesian**: Shiryaev’61, Borovkov’98, Tartakovsky&Veeravalli’05.
- **Minimax**: CUSUM (Page’54, Lorden’71).
Application in Cognitive Radio

- **Measurements:** \(\{X_1, X_2, \ldots, X_{T_0-1}\}\) are i.i.d with distribution \(f_0(x)\);
  \(\{X_{T_0}, X_{T_0+1}, \ldots\}\) are i.i.d with distribution \(f_1(x)\).

- **Stopping Time:** At time \(t = T_d\), the user declares an opportunity.

- **Quickest Detection:** \(\min \mathbb{E}[\underbrace{(T_d - T_0)^+}_{\text{Detection Delay}}]\) subject to \(\Pr[\underbrace{T_d < T_0}_{\text{Interference Constraint}}] \leq \zeta\).
Quickest Detection in Multiple On-Off Processes

► Quickest Detection of Idle Periods in Multiple On-Off Processes:
  □ Continue, switch, or declare?

► Tradeoffs:
  □ Whether to declare: delay vs. reliability.
  □ Whether to switch: loss of data vs. avoiding bad realizations.
Climbing The Corporate Ladder

Keep climbing?

Or look for a greener pasture?
Outline

► Quickest change detection in a single stochastic process
  □ Shiryaev’s algorithm

► Quickest detection in multiple on-off processes
  □ A decision-theoretic formulation
  □ The optimal detection rule: a threshold policy

► Simulation examples

► Conclusion and work in progress
Quickest Change Detection: Classic Bayesian Formulation

Bayesian Formulation:
- Prior distribution of change point $T_0$: geometric

\[
\begin{align*}
\Pr[T_0 = 0] &= \lambda_0 \\
\Pr[T_0 = k] &= (1 - \lambda_0)p(1 - p)^{k-1}, \quad \forall k > 0,
\end{align*}
\]
Shiryaev’s Algorithm

Change Point $T_0$  
Declare at $T_d$

Detection Delay

$t$

$X_1$ $X_2$ $\cdots$ $X_{T_0-1}$ $X_{T_0}$ $X_{T_0+1}$ $\cdots$ $X_{T_d}$ $\cdots$

i.i.d. $\sim f_0(x)$  
i.i.d. $\sim f_1(x)$

- A sufficient statistic: a posterior probability that change has occurred
  \[ \lambda_t \triangleq Pr[T_0 \leq t | X_1, X_2, \ldots, X_t]. \]

- Shiryaev’s detection rule:
  \[ T_d = \inf\{t : \lambda_t \geq \eta_d\} \]

- Detection threshold $\eta_d$: determined by the reliability constraint $\zeta$.

- Setting $\eta_d = 1 - \zeta$ is asymptotically optimal as $\zeta \to 0$. 
Quickest Detection In Multiple On-Off Processes

- A large number of independent homogeneous on-off processes.
- Busy period: geometrically distributed with mean $m_B = \frac{1}{p_B}$.
- Idle period: geometrically distributed with mean $m_I = \frac{1}{p_I}$.
- Fraction of idle time: $\lambda_0 = \frac{m_I}{m_I + m_B}$. 
Quickest Detection In Multiple On-Off Processes

\[
\min \mathbb{E}\left[ \sum_{l=1}^{L-1} T_s(l) + T_d(L) \right] \quad \text{s.t.} \quad \Pr\left[ \sum_{l=1}^{L-1} T_s(l) + T_d(L) = \text{busy} \right] \leq \zeta
\]

Detection Time

Reliability Constraint
A POMDP Formulation

▶ State Transition:

![Diagram of state transitions]

- **0 (Busy)**
  - \( S/(1 - \lambda_0) \)
  - \( C/1 - p_B \)

- **1 (Idle)**
  - \( S/\lambda_0 \)
  - \( C/1 - p_I \)

- **Δ (Absorbing)**
  - \( D/1 \)

▶ Cost:

- Switch or Continue: \( 1 \)
- Declare during a busy period: \( \gamma \)
A POMDP Formulation

- **A Sufficient Statistic:** the information state (belief)

\[\lambda_t = \Pr[Z_t = \text{idle}|X_1, X_2, \ldots, X_t] \]

\[\lambda_0 = \frac{m_I}{m_I + m_B} \]

- **Update of the Information State**

\[\lambda_t = \begin{cases} 
    \mathcal{T}(\lambda_0|x) & a(t-1) = S, \ X_t = x \\
    \mathcal{T}(\lambda_{t-1}|x) & a(t-1) = C, \ X_t = x \end{cases} \]

- **\( \mathcal{T}(\lambda|x): \)** updated information state based on the new measurement \( x \).

\[\mathcal{T}(\lambda|x) \overset{\Delta}{=} \frac{(\lambda p_I + \lambda p_B)f_1(x)}{(\lambda p_I + \lambda p_B)f_1(x) + (\lambda p_I + \lambda p_B)f_0(x)}.\]
A POMDP Formulation

- Channel switching and change detection policy \( \pi \):

\[
\lambda_t \in [0, 1] \quad \Rightarrow \quad a(t) \in \{S, C, D\}, \text{ for each time } t.
\]

- Quickest change detection:

\[
\pi^* = \arg \min_{\pi} \mathbb{E}_{\pi} \left[ \sum_{t=0}^{\infty} R_{\pi}(\lambda_t) \mid \lambda_0 = \frac{m_I}{m_B + m_I} \right],
\]

\text{Cost}
Quickest Change Detection: Value Functions

- **$V(\lambda_t)$**: the minimum expected total cost when the current info. state is $\lambda_t$.

  $$V(\lambda_t) = \min \{ V_C(\lambda_t), \ V_S(\lambda_t), \ V_D(\lambda_t) \}.$$ 

- **$V_C(\lambda_t)$**: the minimum expected total cost if continue.

  $$V_C(\lambda_t) = 1 + \int_x \underbrace{P(x; \lambda_t)}_{\Pr[\text{observe } x \text{ under } \lambda_t]} \underbrace{V(T(\lambda_t|x))}_{\text{d}x}$$

- **$V_S(\lambda_t)$**: the minimum expected total cost if switch.

  $$V_S(\lambda_t) = 1 + \int_x \underbrace{P(x; \lambda_0)}_{\Pr[\text{observe } x \text{ under } \lambda_0]} \underbrace{V(T(\lambda_0|x))}_{\text{d}x} = V_C(\lambda_0)$$

- **$V_D(\lambda_t)$**: the minimum expected total cost if declare.

  $$V_D(\lambda_t) = (1 - \lambda_t) \gamma.$$
Quickest Change Detection: A Threshold Policy

- $V_D(\lambda_t)$ is linear.
- $V_C(\lambda_t)$ is concave and monotonically decreasing if $\frac{1}{m_B} + \frac{1}{m_I} \leq 1$.
- $V_S(\lambda_t) = V_C(\lambda_0)$, where $\lambda_0 = \frac{m_I}{m_I + m_B}$.
Quickest Change Detection: A Threshold Policy

\[ V(\lambda_t) = \min \{ V_S(\lambda_t), V_C(\lambda_t), V_D(\lambda_t) \}. \]

Switch, Continue, Declare
Simulation Examples

- $f_0(x), f_1(x)$: Gaussian with zero mean and different variances.

- SNR = 10dB.

- $\eta_d = 1 - \zeta$. 
Simulation Example: Geometric Distribution

- Increase both $m_B$ and $m_I$ while keeping $\lambda_0$ fixed
Simulation Example: Arbitrary Distributions

- Busy period: truncated Pareto distribution with increasing tail index
Conclusion and Work in Progress

Quickest Detection in Multiple On-Off Processes:

Work in Progress:

- Asymptotic optimality for arbitrary distributions and non-i.i.d. data.
- Minimax formulation.